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Franz Barthelmes

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Theory and formulas used by the calculation service of the International Centre for Global Earth Models (ICGEM)
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Contents

1 Introduction

The intention of this article is to present the definitions of different functionals of the Earth's gravity field and possibilities for their approximative calculation from a mathematical representation of the outer potential. In history this topic has usually been treated in connection with the boundary value problems of geodesy, i.e. starting from measurements at the Earth's surface and their use to derive a mathematical representation of the geopotential.

Nowadays global gravity field models, mainly derived from satellite measurements, become more and more detailed and accurate and, additionally, the global topography can be determined by modern satellite methods independently from the gravity field. On the one hand the accuracy of these gravity field models has to be evaluated and on the other hand they should be combined with classical (e.g. gravity anomalies) or recent (e.g. GPS-levelling-derived or altimetry-derived geoid heights) data. Furthermore, an important task of geodesy is to make the gravity field functionals available to other geosciences. For all these purposes it is necessary to calculate the corresponding functionals as accurately as possible or, at least, with a well-defined accuracy from a given global gravity field model and, if required, with simultaneous consideration of the topography model.

We will start from the potential, formulate the definition of some functionals and derive the formulas for the calculation. In doing so we assume that the Earth's gravity potential is known outside the masses, the normal potential outside the ellipsoid and that mathematical representations are available for both. Here we neglect time variations and deal with the stationary part of the potential only.

Approximate calculation formulas with different accuracies are formulated and specified for the case that the mathematical representation of the potential is in terms of spherical harmonics. The accuracies of the formulas are demonstrated by practical calculations using the gravity field model EIGEN-GL04C (Förste et al., 2006).

More or less, what is compiled here is well-known in physical geodesy but distributed over a lot of articles and books which are not cited here. In the first instance this text is targeted at non-geodesists and it should be "stand-alone readable".

Textbooks for further study of physical geodesy are (Heiskanen & Moritz, 1967; Pick et al., 1973; Vaníček & Krakiwsky, 1982; Torge, 1991; Moritz, 1989; Hofmann-Wellenhof & Moritz, 2005).

2 Definitions

2.1 The Potential and the Geoid

As it is well-known, according to Newton's law of gravitation, the potential W_a of an attractive body with mass density ρ is the integral (written in cartesian coordinates x, y, z)

$$\boxed{W_a(x,y,z) = G \iiint\limits_v \frac{\rho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} dx'dy'dz'} \tag{1}$$

over the volume v of the body, where G is the Newtonian gravitational constant, and $dv = dx'dy'dz'$ is the element of volume. For $\sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \to \infty$ the potential W_a behaves like the potential of a point mass located at the bodies centre of mass with the total mass of the body. It can be shown that W_a satisfies *Poisson's equation*

$$\nabla^2 W_a = -4\pi G\rho \tag{2}$$

where ∇ is the Nabla operator and ∇^2 is called the Laplace operator (e.g. Bronshtein et al., 2004). Outside the masses the density ρ is zero and W_a satisfies *Laplace's equation*

$$\nabla^2 W_a = 0 \tag{3}$$

thus W_a is a *harmonic function* in empty space (e.g. Blakely, 1995).

On the rotating Earth, additionally to the attracting force, also the centrifugal force is acting which can be described by its (non-harmonic) centrifugal potential

$$\Phi(x, y, z) = \frac{1}{2}\,\omega^2 d_z^2 \tag{4}$$

where ω is the angular velocity of the Earth and $d_z = \sqrt{x^2 + y^2}$ is the distance to the rotational (z-) axis. Hence, the potential W associated with the rotating Earth (e.g. in an Earth-fixed rotating coordinate system) is the sum of the attraction potential W_a and the centrifugal potential Φ

$$W = W_a + \Phi \tag{5}$$

The associated force vector $\vec{g}$ acting on a unit mass, the *gravity vector*, is the gradient of the potential

$$\vec{g} = \nabla W \tag{6}$$

and the magnitude

$$g = |\nabla W| \tag{7}$$

is called *gravity*. Potentials can be described (and intuitively visualised) by its equipotential surfaces. From the theory of harmonic functions it is known, that the knowledge of one equipotential surface is sufficient to define the whole harmonic function outside this surface.

For the Earth one equipotential surface is of particular importance: the *geoid*. Among all equipotential surfaces, the geoid is the one which coincides with the undisturbed sea surface (i.e. sea in static equilibrium) and its fictitious continuation below the continents as sketched in Fig. 1 (e.g. Vaníček & Christou, 1994, Vaníček & Krakiwsky, 1982 or Hofmann-Wellenhof & Moritz, 2005). Being an equipotential sur-

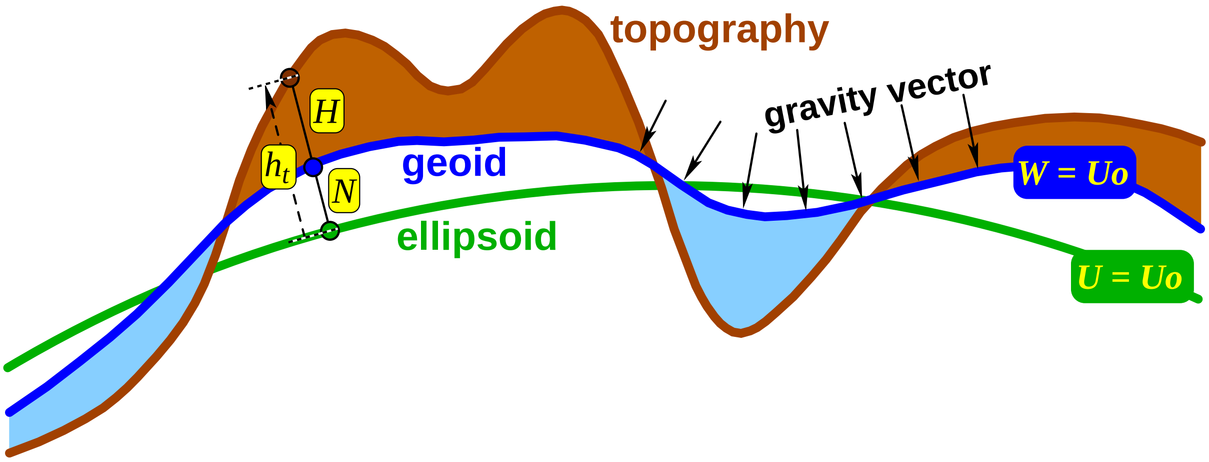


Figure 1: The ellipsoid, the geoid and the topography

face, the geoid is a surface to which the force of gravity is everywhere perpendicular (but not equal in magnitude!). To define the geoid surface in space, simply the correct value W_0 of the potential has to be chosen:

$$\boxed{W(x, y, z) = W_0 = constant} \tag{8}$$

As usual we split the potential W into the normal potential U and the disturbing potential T

$$W(x, y, z) = U(x, y, z) + T(x, y, z) \tag{9}$$

and define "shape" and "strengths" of the normal potential as follows: (a) The equipotential surfaces ($U(x, y, z) = constant$) of the normal potential should have the shapes of ellipsoids of revolution and (b) the equipotential surface for which holds $U(x, y, z) = W_0$ (see eq. 8) should approximate the geoid, i.e. the undisturbed sea surface, as good as possible (i.e. in a least squares fit sense). It is advantageous to define ellipsoidal coordinates (h, λ, ϕ) with respect to this level ellipsoid $U(h = 0) = U_0 = W_0$, where h is the height above ellipsoid (measured along the ellipsoidal normal), λ is the ellipsoidal longitude and ϕ the ellipsoidal latitude. Thus eq. (9) writes (note that the normal potential U does not depend on λ):

$$\boxed{W(h, \lambda, \phi) = U(h, \phi) + T(h, \lambda, \phi)} \tag{10}$$

and the geoid, in ellipsoidal coordinates, is the equipotential surface for which holds

$$\boxed{W\big(h = N(\lambda, \phi), \lambda, \phi\big) = U\big((h = 0), \phi\big) = U_0} \tag{11}$$

where $N(\lambda, \phi)$ is the usual representation of the geoid as heights N with respect to the ellipsoid ($U = U_0$) as a function of the coordinates λ and ϕ. Thus N are the undulations of the geoidal surface with respect to the ellipsoid. This geometrical ellipsoid together with the normal ellipsoidal potential is called *Geodetic Reference System* (e.g. NIMA, 2000 or Moritz, 1980). Now, with the ellipsoid and the geoid, we have two reference surfaces with respect to which the height of a point can be given. We will denote the height of the Earth's surface, i.e. the height of the topography, with respect to the ellipsoid by h_t, and with respect to the geoid by H, hence it is (see fig. 1):

$$\boxed{h_t(\lambda, \phi) = N(\lambda, \phi) + H(\lambda, \phi)} \tag{12}$$

Here H is assumed to be measured along the ellipsoidal normal and not along the real plumb line, hence it is not exactly the orthometric height. A discussion of this problem can be found in (Jekeli, 2000).

Like the potential W (eq. 5) the normal potential also consists of an attractive part U_a and the centrifugal potential Φ

$$U = U_a + \Phi \tag{13}$$

and obviously, the disturbing potential

$$T(h, \lambda, \phi) = W_a(h, \lambda, \phi) - U_a(h, \phi) \tag{14}$$

does not contain the centrifugal potential and is harmonic outside the masses. The gradient of the normal potential

$$\vec{\gamma} = \nabla U \tag{15}$$

is called *normal gravity vector* and the magnitude

$$\gamma = |\nabla U| \tag{16}$$

is the *normal gravity*.

For functions generated by mass distributions like W_a from eq. (1), which are harmonic outside the masses, there are harmonic or analytic continuations W_a^c which are equal to W_a outside the masses and are (unlike W_a) also harmonic inside the generating masses. But the domain where W_a is harmonic, i.e. satisfies Laplace's equation (eq. 3), can not be extended completely into its generating masses because there must be singularities somewhere to generate the potential, at least, as is well known, at one point at the centre if the mass distribution is spherically symmetric. How these singularities look like (point-, line-, or surface-singularities) and how they are distributed depends on the structure of the function W_a outside the masses, i.e. (due to eq. 1) on the density distribution ρ of the masses. Generally one can say that these singularities are deeper, e.g. closer to the centre of mass, if the potential W_a is "smoother". For further study of this topic see (Zidarov, 1990) or (Moritz, 1989).

Due to the fact that the height $H = h_t - N$ of the topography with respect to the geoid is small compared to the mean radius of the Earth and that in practise the spatial resolution (i.e. the roughness) of the approximative model for the potential W_a will be limited (e.g. finite number of coefficients or finite number of sampling points), we expect that the singularities of the downward continuation of W_a lie deeper than the geoid and assume that W_a^c exists without singularities down to the geoid so that we can define (h_t is the ellipsoidal height of the Earth's surface, see eq. 12):

$$\boxed{\begin{array}{rcll} W_a^c(h,\lambda,\phi) & = & W_a(h,\lambda,\phi) & \text{for} \quad h \geq h_t \\ \nabla^2 W_a^c & = & 0 & \text{for} \quad h \geq \min(N,h_t) \\ W^c(h,\lambda,\phi) & = & W_a^c(h,\lambda,\phi) + \Phi(h,\phi) & \end{array}} \tag{17}$$

However, this can not be guaranteed and has to be verified, at least numerically, in practical applications.

From its definition the normal potential U_a is harmonic outside the normal ellipsoid and it is known that a harmonic downward continuation U_a^c exists down to a singular disk in the centre of the flattened rotational ellipsoid (e.g. Zidarov, 1990). Thus, downward continuation of the normal potential is no problem and we can define

$$\boxed{\begin{array}{rcll} U_a^c(h,\phi) & = & U_a(h,\phi) & \text{for} \quad h \geq 0 \\ \nabla^2 U_a^c & = & 0 & \text{for} \quad h \geq \min(N,0,h_t) \\ U^c(h,\phi) & = & U_a^c(h,\phi) + \Phi(h,\phi) & \end{array}} \tag{18}$$

and hence

$$\boxed{\begin{array}{rcll} T^c(h,\lambda,\phi) & = & W_a^c(h,\lambda,\phi) - U_a^c(h,\phi) & \\ \nabla^2 T^c & = & 0 & \text{for} \quad h \geq \min(N,h_t) \end{array}} \tag{19}$$

2.2 The Height Anomaly

The *height anomaly* $\zeta(\lambda,\phi)$, the well known approximation of the geoid undulation according to Molodensky's theory, can be defined by the distance from the Earth's surface to the point where the normal potential U has the same value as the geopotential W at the Earth's surface (Molodensky et al., 1962; Hofmann-Wellenhof & Moritz, 2005; Moritz, 1989):

$$\boxed{W(h_t,\lambda,\phi) = U(h_t - \zeta,\lambda,\phi)} \tag{20}$$

where h_t is the ellipsoidal height of the Earth's surface (eq. 12). An illustration of the geometrical situation is given in fig. (2). The surface with the height $\zeta = \zeta(\lambda,\phi)$ with respect to the ellipsoid is often called *quasigeoid* (not shown in fig. 2) and the surface $h_t - \zeta$ is called *telluroid*. It should be emphasised, that the quasigeoid has no physical meaning but is an approximation of the geoid as we will see. In areas where $h_t = N$ (or $H = 0$) i.e. over sea, the quasigeoid coincides with the geoid as can be seen easily from the definition in eq. (20) if we use eq. (12):

$$W(N+H,\lambda,\phi) = U(N+H-\zeta,\lambda,\phi) \tag{21}$$

set $H = 0$ and get

$$W(N,\lambda,\phi) = U(N-\zeta,\phi) \tag{22}$$

and use eq. (11), the definition of the geoid to write

$$U(0,\phi) = U(N-\zeta,\phi) \tag{23}$$

from which follows

$$\boxed{N = \zeta \quad \text{for} \quad H = 0} \tag{24}$$

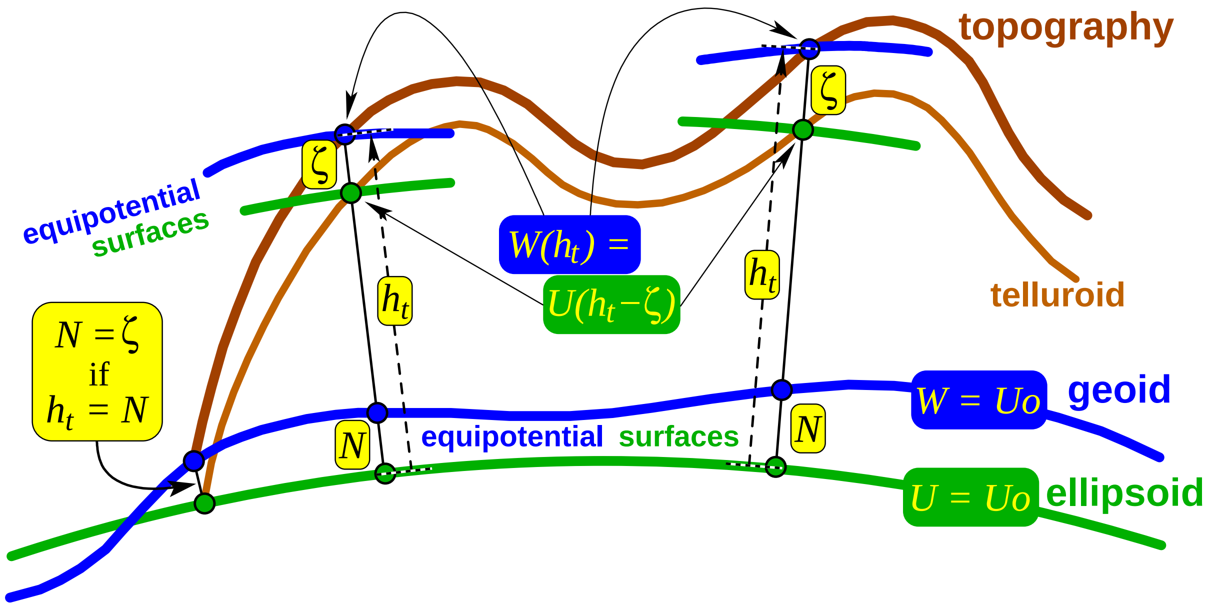


Figure 2: The ellipsoid, the geoid, and the height anomaly ζ

In the history of geodesy the great importance of the height anomaly was that it can be calculated from gravity measurements carried out at the Earth's surface without knowledge of the potential inside the masses, i.e. without any hypothesis about the mass densities.

The definition of eq. (20) is not restricted to heights $h = h_t$ on the Earth's surface, thus a generalised height anomaly $\zeta_g = \zeta_g(h, \lambda, \phi)$ for arbitrary heights h can be defined by:

$$\boxed{W(h, \lambda, \phi) = U(h - \zeta_g, \phi)} \tag{25}$$

2.3 The Gravity Disturbance

The gradient of the disturbing potential T is called the *gravity disturbance vector* and is usually denoted by $\vec{\delta g}$:

$$\vec{\delta g}(h, \lambda, \phi) = \nabla T(h, \lambda, \phi) = \nabla W(h, \lambda, \phi) - \nabla U(h, \phi) \tag{26}$$

The *gravity disturbance* δg is *not* the magnitude of the gravity disturbance vector (as one could guess) but defined as the difference of the magnitudes (Hofmann-Wellenhof & Moritz, 2005):

$$\boxed{\delta g(h, \lambda, \phi) = \big|\nabla W(h, \lambda, \phi)\big| - \big|\nabla U(h, \phi)\big|} \tag{27}$$

In principle, herewith δg is defined for any height h if the potentials W and U are defined there. Additionally, with the downward continuations W_a^c and U_a^c (eqs. 17 and 18), we can define a "harmonic downward continued" gravity disturbance

$$\delta g^c(h, \lambda, \phi) = \big|\nabla W^c(h, \lambda, \phi)\big| - \big|\nabla U^c(h, \phi)\big| \tag{28}$$

With the notations from eqs. (7) and (16) we can write the gravity disturbance in its common form:

$$\boxed{\delta g(h, \lambda, \phi) = g(h, \lambda, \phi) - \gamma(h, \phi)} \tag{29}$$

The reason for this definition is the practical measurement process, where the gravimeter measures only $|\nabla W|$, the magnitude of the gravity, and not the direction of the plumb line.

2.4 The Gravity Anomaly

The term *gravity anomaly* is used with numerous different meanings in geodesy and geophysics and, moreover, there are different practical realisations (cf. Hackney & Featherstone, 2003). Here we will confine ourselves to the *classical free air gravity anomaly*, to the *gravity anomaly according to Molodensky's theory* and to the *topography-reduced gravity anomaly.*

2.4.1 The Classical Definition

The classical (historical) definition in geodesy is the following (cf. Hofmann-Wellenhof & Moritz, 2005): The gravity anomaly Δg_{cl} (subscript "cl" stands for "classical") is the magnitude of the downward continued gravity $|\nabla W^c|$ (eq. 17) onto the *geoid* minus the normal gravity $|\nabla U|$ on the *ellipsoid* at the same ellipsoidal longitude λ and latitude ϕ:

$$\boxed{\Delta g_{cl}(\lambda,\phi) = \left|\nabla W^c(N,\lambda,\phi)\right| - \left|\nabla U(0,\phi)\right|} \tag{30}$$

The origin of this definition is the (historical) geodetic practise where the altitude of the gravity measurement was known only with respect to the geoid from levelling but not with respect to the ellipsoid. The geoid height N was unknown and should be determined just by these measurements. The classical formulation of this problem is the *Stokes' integral* (e.g. Hofmann-Wellenhof & Moritz, 2005; Martinec, 1998). For this purpose the measured gravity $|\nabla W(h_t,\lambda,\phi)|$ has to be reduced somehow down onto the geoid and the exact way to do so is the harmonic downward continuation of the attraction potential W_a (eq. 17). This is the reason for the definition of the classical gravity anomaly in eq. (30). In practise the so-called "free air reduction" has been or is used to get $|\nabla W^c(N,\lambda,\phi)|$ approximately. Thus the classical gravity anomaly depends on longitude and latitude only and is not a function in space.

2.4.2 The Modern Definition

The generalised gravity anomaly Δg according to Molodensky's theory (Molodensky et al., 1962; Hofmann-Wellenhof & Moritz, 2005; Moritz, 1989) is the magnitude of the gravity at a given point (h,λ,ϕ) minus the normal gravity at the same ellipsoidal longitude λ and latitude ϕ but at the ellipsoidal height $h-\zeta_g$, where ζ_g is the generalised height anomaly from definition (25):

$$\boxed{\Delta g(h,\lambda,\phi) = \left|\nabla W(h,\lambda,\phi)\right| - \left|\nabla U(h-\zeta_g,\phi)\right|, \quad \text{for} \quad h \ge h_t} \tag{31}$$

or in its common form:

$$\boxed{\Delta g(h,\lambda,\phi) = g(h,\lambda,\phi) - \gamma(h-\zeta_g,\phi)} \tag{32}$$

Here the height h is assumed on or outside the Earth's surface, i.e. $h \ge h_t$, hence with this definition the gravity anomaly is a function in the space outside the masses. The advantage of this definition is that the measured gravity $|\nabla W|$ at the Earth's surface can be used without downward continuation or any reduction. If geodesists nowadays speak about gravity anomalies, they usually have in mind this definition with $h = h_t$, i.e. on the Earth's surface.

2.4.3 The Topography-Reduced Gravity Anomaly

For many purposes a functional of the gravitational potential is needed which is the difference between the real gravity and the gravity of the reference potential and which, additionally, does not contain the effect of the topographical masses above the geoid. The well-known "Bouguer anomaly" or "Refined Bouguer anomaly" (e.g. Hofmann-Wellenhof & Moritz, 2005) are commonly used in this connection. However, they are defined by reduction formulas and not as functionals of the potential. The problems arising when using the concepts of the Bouguer plate or the Bouguer shell are discussed in (Vaníček et al., 2001) and (Vaníček et al., 2004).

Thus, let us define the gravity potential of the topography V_t, i.e. the potential induced by all masses lying above the geoid. Analogously to eq. (27), we can now define a gravity disturbance δg_{tr} which does not contain the gravity effect of the topography:

$$\boxed{\delta g_{tr}(h,\lambda,\phi) = \left|\nabla\left[W(h,\lambda,\phi) - V_t(h,\lambda,\phi)\right]\right| - \left|\nabla U(h,\phi)\right|} \tag{33}$$

and, analogously to eq. (31), a *topography-reduced gravity anomaly* Δg_{tr}:

$$\boxed{\Delta g_{tr}(h,\lambda,\phi) = \left|\nabla\left[W(h,\lambda,\phi) - V_t(h,\lambda,\phi)\right]\right| - \left|\nabla U(h-\zeta,\phi)\right|} \tag{34}$$

where, consequently, $W - V_t$ is the gravity potential of the Earth without the masses above the geoid.

The difficulty here is, that the potential V_t (or it's functionals) cannot be measured directly but can only be calculated approximately by using a digital terrain model of the whole Earth and, moreover, a hypothesis about the density distribution of the masses.

Approximate realisations of such anomalies are mainly used in geophysics and geology because they show the effects of different rock densities of the subsurface. If geophysicists or geologists speak about gravity anomalies they usually have in mind this type of anomalies.

3 Approximation and Calculation

3.1 The Geoid

As one can see from the definition in eq. (8) or eq. (11), the calculation of the geoid is the (iterative) search of all points in space which have the same gravity potential $W = W_0 = U_0$. Let us assume that the geopotential $W(h,\lambda,\phi)$ is known also inside the masses and $N_i(\lambda,\phi)$ is a known approximative value for the exact geoid height $N(\lambda,\phi)$ (e.g. as result of the i-th step of an iterative procedure). Here we should have in mind that the representation N of the geoidal surface is with respect to the normal ellipsoid which already is a good approximation of the geoid in the sense that the biggest deviations of the geoid from the ellipsoid with respect to its semi-major axis is in the order of 10^{-5}.

The difference $W(N) - W(N_i)$ for the coordinates λ and ϕ is (approximately):

$$W(N) - W(N_i) \approx (N - N_i) \cdot \left.\frac{\partial W}{\partial h}\right|_{h=N_i} \tag{35}$$

The ellipsoidal elevation h is taken along the ellipsoidal normal which is given by the negative direction of the gradient of the normal potential $-\frac{\nabla U}{|\nabla U|}$. Thus the partial derivative $\frac{\partial W}{\partial h}$ can be represented by the normal component of the gradient ∇W, i.e. by the projection onto the *normal plumb line* direction

$$-\frac{\partial W}{\partial h} = \left\langle \frac{\nabla U}{|\nabla U|} \middle| \nabla W \right\rangle \tag{36}$$

or, because the directions of ∇W and ∇U nearly coincide, by:

$$-\frac{\partial W}{\partial h} \approx \left\langle \frac{\nabla W}{|\nabla W|} \middle| \nabla W \right\rangle = \left|\nabla W\right| \tag{37}$$

where $\langle \vec{a} \,|\, \vec{b} \rangle$ denotes the scalar product of the vectors $\vec{a}$ and $\vec{b}$ and, if $\vec{a}$ has the unit length as in eqs. (36) and (37), the projection of $\vec{b}$ onto the direction of $\vec{a}$. By replacing $W(N)$ by U_0 according to eq. (11) and with the notation $g = |\nabla W|$ (eq. 7) we can write

$$U_0 - W(N_i) \approx -g(N_i)\,(N - N_i) \tag{38}$$

for eq. (35) and thus the geoid height N can (approximately) be calculated by

$$N \approx N_i + \frac{1}{g(N_i)} \big[W(N_i) - U_0 \big] \tag{39}$$

and the reasons for "$\approx$" instead of "$=$" are the linearisation in eq. (35) and the approximation in eq. (37). That means, if the gravity potential W is known also inside the topographic masses, eq. (39) can be used to calculate the geoid iteratively with arbitrary accuracy for each point (λ, ϕ):

$$N_{i+1}(\lambda, \phi) = N_i(\lambda, \phi) + \frac{1}{g(N_i, \lambda, \phi)} \big[W(N_i, \lambda, \phi) - U_0 \big] \tag{40}$$

provided that we have an appropriate starting value for the iteration and the iteration converges. Replacing the gravity $g(N_i)$ (eq. 7) by the normal gravity $\gamma(0)$ (eq. 16) will not change the behaviour of this iteration, because each step will be scaled only by a factor of $(1 - g/\gamma)$, which is in the order of 10^{-4} or smaller. So we write:

$$\boxed{N_{i+1}(\lambda, \phi) = N_i(\lambda, \phi) + \frac{1}{\gamma(0, \phi)} \big[W(N_i, \lambda, \phi) - U_0 \big]} \tag{41}$$

With $i = 0$ and $N_0 = 0$ in eq. (41) we get:

$$N_1(\lambda, \phi) = \frac{1}{\gamma(0, \phi)} \big[W(0, \lambda, \phi) - U_0 \big] \tag{42}$$

and with $W(0) = U_0 + T(0)$ (eq. 10) we finally have

$$\boxed{N_1(\lambda, \phi) = \frac{T(0, \lambda, \phi)}{\gamma(0, \phi)}} \tag{43}$$

as a first approximate value for $N(\lambda, \phi)$ which is the well-known Bruns' formula (e.g. Hofmann-Wellenhof & Moritz, 2005).

To get an estimation of the difference $N_2 - N_1$ from eq. (41) we write:

$$N_2(\lambda, \phi) - N_1(\lambda, \phi) = \frac{1}{\gamma(0, \phi)} \big[W(N_1, \lambda, \phi) - U_0 \big] \tag{44}$$

and replace again W by $U + T$ (eq. 10) and get:

$$N_2(\lambda, \phi) - N_1(\lambda, \phi) = \frac{1}{\gamma(0, \phi)} \big[U(N_1, \lambda, \phi) + T(N_1, \lambda, \phi) - U_0 \big] \tag{45}$$

With the linearisation

$$U(N_1) \approx U(0) + N_1 \cdot \left. \frac{\partial U}{\partial h} \right|_{h=0} \tag{46}$$

and the notation (cf. eq. 16)

$$\gamma = \big| \nabla U \big| = \left\langle \frac{\nabla U}{|\nabla U|} \middle| \nabla U \right\rangle = -\frac{\partial U}{\partial h} \tag{47}$$

we get

$$N_2(\lambda, \phi) - N_1(\lambda, \phi) \approx \frac{1}{\gamma(0, \phi)} \big[-\gamma(0, \phi) N_1 + T(N_1, \lambda, \phi) \big] \tag{48}$$

Replacing N_1 on the right hand side by Bruns' formula (eq. 43) we get

$$N_2(\lambda, \phi) - N_1(\lambda, \phi) \approx \frac{T(N_1, \lambda, \phi) - T(0, \lambda, \phi)}{\gamma(0, \phi)} \tag{49}$$

for the difference, and for N_2:

$$N_2(\lambda, \phi) \approx \frac{T(N_1, \lambda, \phi)}{\gamma(0, \phi)} \tag{50}$$

The difference $T(N_1) - T(0)$ in eq. (49) can be approximated by

$$T(N_1) - T(0) \approx N_1 \cdot \left.\frac{\partial T}{\partial h}\right|_{h=0} \tag{51}$$

and we get

$$\boxed{N_2 - N_1 \approx N_1 \cdot \frac{1}{\gamma(0)} \left.\frac{\partial T}{\partial h}\right|_{h=0}} \tag{52}$$

The factor on the right hand side which scales N_1 is in the order of 10^{-4} or smaller, i.e. $N_2 - N_1$ is in the order of some millimetres. That means we can expect (if eq. 41 converges fast, i.e. if the step size decays rapidly) that N_1 is a good approximation of N and with eq. (50) we can define

$$\boxed{\tilde{N}_2 = \frac{T(N_1, \lambda, \phi)}{\gamma(0, \phi)} \approx N_1 \left(1 + \frac{1}{\gamma(0)} \left.\frac{\partial T}{\partial h}\right|_{h=0}\right)} \tag{53}$$

which should be even better.

Usually we don't know the potential inside the masses, therefore we will do the following: We replace W_a, the attraction part of W, by its harmonic downward continuation W_a^c (eq. 17) and thus T by T^c (eq. 19) and compute an associated geoid height $N^c(\lambda, \phi)$ which is also an approximation of the real geoid height N. Then we try to calculate (approximately) the difference $N - N^c$ caused by the masses above the geoid. Analogous to the iterative calculation of the geoid N from the potential W by eq. (41) the calculation of N^c from W^c writes:

$$\boxed{N_{i+1}^c(\lambda, \phi) = N_i^c(\lambda, \phi) + \frac{1}{\gamma(0, \phi)} \left[W^c(N_i^c, \lambda, \phi) - U_0\right]} \tag{54}$$

Obviously eqs. (43) to (53) are also valid for the harmonic downward continued potential T^c instead of T, thus N_1 and N_1^c or even better $\tilde{N}_2$ and $\tilde{N}_2^c$ are good approximations for N and N^c respectively:

$$\boxed{N_1^c(\lambda, \phi) = \frac{T^c(0, \lambda, \phi)}{\gamma(0, \phi)}} \tag{55}$$

$$\boxed{\tilde{N}_2^c(\lambda, \phi) = \frac{T^c(N_1^c, \lambda, \phi)}{\gamma(0, \phi)} \approx N_1^c \left(1 + \frac{1}{\gamma(0)} \left.\frac{\partial T^c}{\partial h}\right|_{h=0}\right)} \tag{56}$$

The convergence behaviour of the iterative solution in eq. (41) or (54) will not be discussed in (theoretical) detail here. However, if we consider that the maximum relative differences between the ellipsoidal normal potential U and the real potential W are in the order of 10^{-5} (as mentioned above) a very fast convergence can be expected, which is confirmed by the practical calculations in section (5) (cf. figs. 6 and 7).

To estimate the difference $N - N^c$ between the real geoid and the approximated geoid using the downward continued potential we use $\tilde{N}_2$ and $\tilde{N}_2^c$ from eqs. (53) and (56) and write

$$N(\lambda, \phi) - N^c(\lambda, \phi) \approx \frac{1}{\gamma(0, \phi)} \left[T(N_1, \lambda, \phi) - T^c(N_1^c, \lambda, \phi)\right] \approx \frac{1}{\gamma(0, \phi)} \left[T(N, \lambda, \phi) - T^c(N, \lambda, \phi)\right] \tag{57}$$

To estimate the difference $T - T^c$ in eq. (57) by using information about the topography we introduce the potential $V_t(h, \lambda, \phi)$, induced by all topographical masses above the geoid, which describes the potential also inside the masses, and the potential $V_t^c(h, \lambda, \phi)$, the harmonic downward continuation of V_t. To get the harmonic downward continuation T^c of the disturbing potential T down to the geoid, we must downward continue only the part of the potential caused by the topographic masses and write

$$T^c(N, \lambda, \phi) = T(N, \lambda, \phi) - V_t(N, \lambda, \phi) + V_t^c(N, \lambda, \phi) \tag{58}$$

Then for eq. (57) we get

$$N(\lambda, \phi) - N^c(\lambda, \phi) \approx \frac{1}{\gamma(0, \phi)} \left[V_t(N, \lambda, \phi) - V_t^c(N, \lambda, \phi) \right] \tag{59}$$

and for the geoid:

$$\boxed{N(\lambda, \phi) \approx N^c(\lambda, \phi) + \frac{1}{\gamma(0, \phi)} \left[V_t(N, \lambda, \phi) - V_t^c(N, \lambda, \phi) \right]} \tag{60}$$

To find an approximation of the difference $V_t - V_t^c$ in eq. (59) let's treat the potential $V_s(r)$ of a spherical mass shell from radius R_1 to R_2, with constant mass density ρ (see fig. 3).

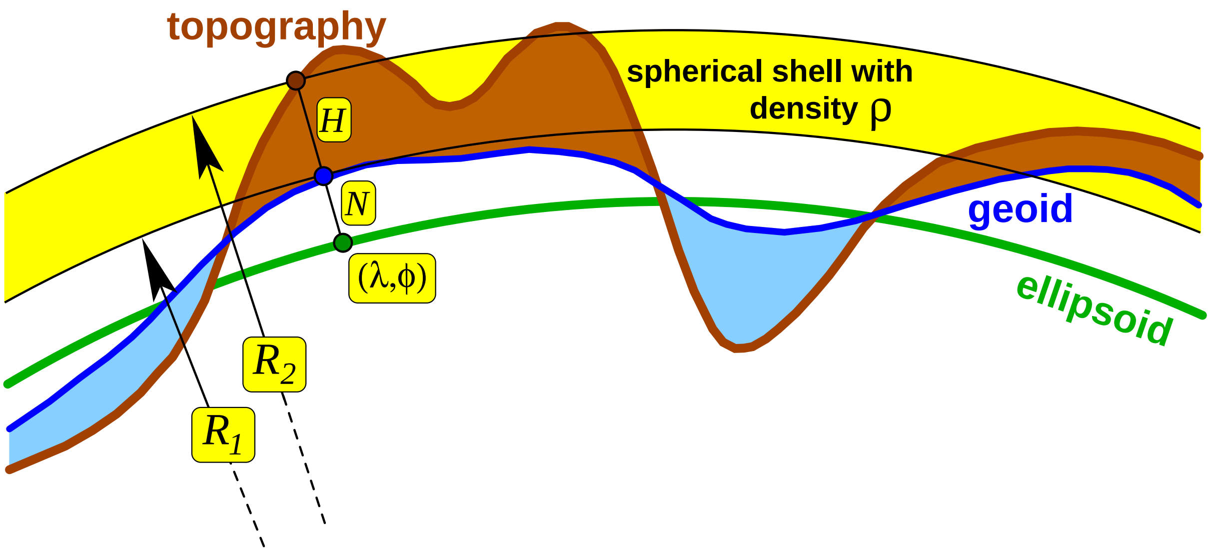


Figure 3: Approximation of the topography at one point (λ, ϕ) by a spherical shell of homogenous mass density

The potential outside the shell is

$$V_s(r) = \frac{GM_s}{r} \quad \text{for} \quad r \geq R_2 \tag{61}$$

where M_s is the total mass of the shell and G is the gravitational constant. For $r \geq R_2$ it is the same potential than that of a point mass with the same total mass M_s located at the origin of the coordinate system. Hence the downward continuation of eq. (61) is simply the same formula defined for smaller values of r:

$$V_s^c(r) = \frac{GM_s}{r} \quad \text{for} \quad r \geq R_1 \tag{62}$$

The mass M_s of the shell is

$$M_s = \frac{4\pi \rho}{3}(R_2^3 - R_1^3) \tag{63}$$

Below the shell (inside the mass free inner sphere) the potential is constant:

$$V_s(r) = 2\pi\, G\rho\, (R_2^2 - R_1^2) \quad \text{for} \quad r \leq R_1 \tag{64}$$

Using this simple spherical shell approximation for the topographical masses with the height H above the geoid at one specific point (λ, ϕ), we get for V_t^c, the downward continued potential on the geoid, from eqs. (62) and (63) with $r = R_1$ and $R_2 = R_1 + H$

$$V_t^c(N) \approx \frac{4\pi\, G\rho}{3} \left(\frac{(R_1 + H)^3 - R_1^3}{R_1} \right) = 4\pi\, G\rho \left(R_1 H + H^2 + \frac{H^3}{3R_1} \right) \tag{65}$$

and for V_t, the potential inside the masses on the geoid, from eq. (64)

$$V_t(N) \approx 2\pi\, G\rho\, \left((R_1 + H)^2 - R_1^2 \right) = 2\pi\, G\rho\, (2R_1 H + H^2) \tag{66}$$

Thus the difference $V_t(N) - V_t^c(N)$ can be approximated by

$$\begin{aligned} V_t(N) - V_t^c(N) &\approx 2\pi\, G\rho \left(2R_1 H + H^2 - 2R_1 H - 2H^2 - \frac{2H^3}{3R_1} \right) \\ &\approx -2\pi\, G\rho \left(H^2 + \frac{2H^3}{3R_1} \right) = -2\pi\, G\rho H^2 \left(1 + \frac{2H}{3R_1} \right) \end{aligned} \tag{67}$$

and if we neglect the second term due to $H \ll R_1$ (in this approximation R_1 is the distance to the Earth's centre and H/R_1 is in the order of 10^{-4}):

$$\boxed{V_t(N) - V_t^c(N) \approx -2\pi\, G\rho H^2} \tag{68}$$

Hence we get the approximation of $N - N^c$ (eq. 59) at the coordinates λ and ϕ to

$$\boxed{N(\lambda, \phi) - N^c(\lambda, \phi) \approx -\frac{2\pi\, G\rho\, H^2(\lambda, \phi)}{\gamma(0, \phi)}} \tag{69}$$

Thus, if we have a mathematical representation of the disturbing potential $T^c(h, \lambda, \phi)$ and if we know the topography, i.e. the height $H(\lambda, \phi)$, we can calculate an approximation N_1^s (superscript "s" stands for "spherical shell approximation") of the geoid height from eq. (55) and eq. (69) by

$$\boxed{N(\lambda, \phi) \approx N_1^s(\lambda, \phi) = N_1^c(\lambda, \phi) - \frac{2\pi\, G\rho\, H^2(\lambda, \phi)}{\gamma(0, \phi)} = \frac{\left[T^c(0, \lambda, \phi) - 2\pi\, G\rho\, H^2(\lambda, \phi)\right]}{\gamma(0, \phi)}} \tag{70}$$

or, with $\tilde{N}_2^c$ (eq. 56) instead of N_1^c (eq. 55), by an approximation $\tilde{N}_2^s$:

$$\boxed{\boxed{N(\lambda, \phi) \approx \tilde{N}_2^s(\lambda, \phi) = \tilde{N}_2^c(\lambda, \phi) - \frac{2\pi\, G\rho\, H^2(\lambda, \phi)}{\gamma(0, \phi)} = \frac{\left[T^c(N_1^c, \lambda, \phi) - 2\pi\, G\rho\, H^2(\lambda, \phi)\right]}{\gamma(0, \phi)}}} \tag{71}$$

To calculate the geoid with high accuracy the use of a more sophisticated model for the potentials of the topography V_t and V_t^c then the simple spherical shell will be inevitable. However, as seen from eq. (59), only the *difference* between the potential of the topography and its downward continuation has to be approximated instead of the potential itself; therefore the approximation in eq. (68) is more realistic than the approximations for V_t and V_t^c themselves.

3.2 The Height Anomaly

The formulas to calculate the height anomaly from the geopotential can be derived similarly to those for the geoid in section 3.1. With a first approximate value ζ_i for ζ and with

$$U(h_t - \zeta) \approx U(h_t - \zeta_i) - (\zeta - \zeta_i) \cdot \left.\frac{\partial U}{\partial h}\right|_{h_t - \zeta_i} \tag{72}$$

(cf. eq. 46) where h_t is the ellipsoidal height of the Earth's surface from eq. (12), we can write for the difference $U(h_t - \zeta) - U(h_t - \zeta_i)$ at point (λ, ϕ) the linearisation

$$U(h_t - \zeta) - U(h_t - \zeta_i) \approx -(\zeta - \zeta_i) \cdot \left.\frac{\partial U}{\partial h}\right|_{h_t - \zeta_i} \tag{73}$$

By replacing $U(h_t - \zeta)$ with $W(h_t)$ according to the definition of the height anomaly in eq. (20), and with eqs. (47) it follows

$$W(h_t) - U(h_t - \zeta_i) \approx \gamma(h_t - \zeta_i) \cdot (\zeta - \zeta_i) \tag{74}$$

resulting in a better approximation ζ_{i+1} given by

$$\zeta_{i+1} = \zeta_i + \frac{1}{\gamma(h_t - \zeta_i)} \big[W(h_t) - U(h_t - \zeta_i)\big] \tag{75}$$

If we replace $\gamma(h_t - \zeta_i)$ by $\gamma(h_t)$, use $W = U + T$ (eq. 10) and the linearisation

$$U(h_t - \zeta_i) \approx U(h_t) + \gamma(h_t)\zeta_i \tag{76}$$

we again, with the start value of $\zeta_0 = 0$ for $i = 0$, get Bruns' formula

$$\boxed{\zeta_1(\lambda, \phi) = \frac{T(h_t, \lambda, \phi)}{\gamma(h_t, \phi)}} \tag{77}$$

but evaluated here for $h = h_t$ instead of $h = 0$ in eq. (43). Consider eq. (19), from which follows: $T^c(h, \lambda, \phi) = T(h, \lambda, \phi), \quad \text{for} \quad h \geq h_t$. The first approximation ζ_{g1} of the generalised height anomaly ζ_g for arbitrary height h as defined in eq. (25) is then:

$$\boxed{\zeta_{g1}(h, \lambda, \phi) = \frac{T(h, \lambda, \phi)}{\gamma(h, \phi)}} \tag{78}$$

An additional approximation without using topography information, i.e. $h_t = 0$, gives:

$$\boxed{\boxed{\zeta_{e1}(\lambda, \phi) = \zeta_{g1}(0, \lambda, \phi) = N_1^c(\lambda, \phi) = \frac{T^c(0, \lambda, \phi)}{\gamma(0, \phi)}}} \tag{79}$$

which is sometimes called "pseudo-height anomaly" calculated on the ellipsoid and is identical with $N_1^c(\lambda, \phi)$, the approximation of the geoid from eq. (55). Using eqs. (77) and (79), the linearisation

$$T(h_t) \approx T(0) + h_t \cdot \frac{\partial T^c}{\partial h} \tag{80}$$

and $\gamma(0)$ instead of $\gamma(h_t)$ an approximation $\tilde{\zeta}_1$ for the height anomaly on the Earth's surface can also be calculated by:

$$\boxed{\tilde{\zeta}_1(\lambda, \phi) \approx \zeta_{e1}(\lambda, \phi) + \left(\frac{h_t}{\gamma(0)} \cdot \left.\frac{\partial T^c}{\partial h}\right|_{h=0}\right)} \tag{81}$$

which is useful for fast practical calculations (see section 5). In the iteration of eq. (75) we (similarly to eqs. 40 and 41) can replace $\gamma(h_t - \zeta_i)$, the normal gravity at the Telluroid, by $\gamma(0)$, the normal gravity at the ellipsoid, and expect that the convergence behaviour of this iteration will not change:

$$\boxed{\zeta_{i+1}(\lambda,\phi) = \zeta_i(\lambda,\phi) + \frac{1}{\gamma(0)}\big[W(h_t,\lambda,\phi) - U(h_t - \zeta_i,\lambda,\phi)\big]} \tag{82}$$

In contrast to eq. (41) for the geoid height, where the value of the normal potential at the ellipsoid U_0 is the nominal value and the iteration searches for the height $h = N$ where the real potential has the same value $W(N) = U_0$, here, in eq. (82), for the height anomaly, the value $W(h_t)$ for the real potential at the Earth's surface is the target value and the iteration searches for the height $h_t - \zeta$ where the normal potential has the same value $U(h_t - \zeta) = W(h_t)$. In both cases one looks for the distance of two points lying on the same normal plumbline and the normal potential at one point must have the same value as the real potential at the other point. But, for the geoid height N it is done near the ellipsoid, and for the height anomaly ζ it is done near the Earth's surface. However, the main part of the difference between N and ζ does not come from the different ellipsoidal height where the two potentials are compared, but from the fact, that for the geoid the real potential has to be evaluated inside the masses for points over the continents (apart from some "exotic" regions) and on top (outside) of the masses for the height anomaly.

Because of the mathematical similarity of the problems one can expect, that the convergence behaviour of the iteration in eq. (82) should be very similar to that of eq. (54) which is confirmed by the numerical investigations in section (5).

3.3 The Difference: Geoid - Height Anomaly

Now we can estimate the difference between the geoid and the height anomaly. Considering eq. (59) and taking $N^c \approx N_1^c = \zeta_{e1}$ from eq. (79) the difference between the geoid and the pseudo-height anomaly (i.e. the height anomaly on the ellipsoid) is:

$$N(\lambda,\phi) - \zeta_e(\lambda,\phi) \approx \frac{1}{\gamma(0,\phi)}\big[V_t(N,\lambda,\phi) - V_t^c(N,\lambda,\phi)\big] \tag{83}$$

With the approximation of eq. (68) for $V_t - V_t^c$ we get:

$$\boxed{N(\lambda,\phi) - \zeta_e(\lambda,\phi) \approx \frac{2\pi G\rho\, H^2(\lambda,\phi)}{\gamma(0,\phi)}} \tag{84}$$

3.4 The Gravity Disturbance

If we have a mathematical representation of the potential $W(h,\lambda,\phi)$, the calculation of $\delta g(h,\lambda,\phi)$ is no problem. Using arbitrary rectangular coordinates (u,v,w) (global cartesian or local moving trihedron) the gradients ∇W and ∇U of the gravity potential W and the normal potential U, i.e. the vectors of the gravity and the normal gravity, at a given point (h,λ,ϕ), are:

$$\nabla W(h,\lambda,\phi) = W_u(h,\lambda,\phi)\,\vec{e}_u + W_v(h,\lambda,\phi)\,\vec{e}_v + W_w(h,\lambda,\phi)\,\vec{e}_w \tag{85}$$

$$\nabla U(h,\lambda,\phi) = U_u(h,\lambda,\phi)\,\vec{e}_u + U_v(h,\lambda,\phi)\,\vec{e}_v + U_w(h,\lambda,\phi)\,\vec{e}_w \tag{86}$$

where W_u, W_v, W_w, U_u, U_v, U_w are the partial derivatives and $\vec{e}_u$, $\vec{e}_v$, $\vec{e}_w$ are the unit vectors pointing in the direction of u, v and w. Consequently the gravity disturbance can be calculated exactly from eq. (27):

$$\boxed{\begin{aligned}\delta g(h,\lambda,\phi) &= \sqrt{\big[W_u(h,\lambda,\phi)\big]^2 + \big[W_v(h,\lambda,\phi)\big]^2 + \big[W_w(h,\lambda,\phi)\big]^2} \\ &- \sqrt{\big[U_u(h,\lambda,\phi)\big]^2 + \big[U_v(h,\lambda,\phi)\big]^2 + \big[U_w(h,\lambda,\phi)\big]^2}\end{aligned}} \tag{87}$$

One approximation possibility is to use the fact, that the directions of the real gravity vector ∇W and the normal gravity vector ∇U nearly coincide. For this purpose we write eq. (27) in the form:

$$\delta g = \left\langle \frac{\nabla W}{|\nabla W|} \,\middle|\, \nabla W \right\rangle - \left\langle \frac{\nabla U}{|\nabla U|} \,\middle|\, \nabla U \right\rangle \tag{88}$$

with the scalar product notation of eq. (36). Approximating the direction of ∇W by ∇U we get:

$$\delta g \approx \left\langle \frac{\nabla U}{|\nabla U|} \,\middle|\, \nabla W - \nabla U \right\rangle \tag{89}$$

and can use the disturbing potential T to write

$$\boxed{\delta g(h,\lambda,\phi) \approx \left\langle \frac{\nabla U(0,\phi)}{|\nabla U(0,\phi)|} \,\middle|\, \nabla T(h,\lambda,\phi) \right\rangle = -\frac{\partial T(h,\lambda,\phi)}{\partial h}} \tag{90}$$

and, if $h < h_t$, for the harmonic downward continuation δg^c:

$$\boxed{\delta g^c(h,\lambda,\phi) \approx -\frac{\partial T^c(h,\lambda,\phi)}{\partial h}} \tag{91}$$

The unit vector $\frac{\nabla U}{|\nabla U|}$ points in direction of the gradient of the normal potential, i.e. it is the normal plumb line direction. Thus, δg is also (at least approximately) the ellipsoidal normal component of the gravity disturbance vector $\vec{\delta g}$ (e.g. Hofmann-Wellenhof & Moritz, 2005).

An additional approximation is to take the direction of the radius of spherical coordinates (r, λ, φ) instead of the ellipsoidal normal and calculate δg for $h = 0$, i.e. on the ellipsoid, using the downward continuation T^c of the disturbing potential:

$$\boxed{\delta g(0,\lambda,\phi) \approx \delta g_{sa}(0,\lambda,\phi) = -\left.\frac{\partial T^c}{\partial r}\right|_{h=0}} \tag{92}$$

3.5 The Gravity Anomaly

3.5.1 The Classical Gravity Anomaly

If we have a mathematical representation of the potential $W(h,\lambda,\phi)$ the calculation of $\Delta g_{cl}(\lambda,\phi)$ from eq. (30) is no problem, however, the geoid height N has to be calculated beforehand. With eqs. (85) and (86) we get:

$$\boxed{\begin{aligned}\Delta g_{cl}(\lambda,\phi) = & \sqrt{\left[W^c_u(N,\lambda,\phi)\right]^2 + \left[W^c_v(N,\lambda,\phi)\right]^2 + \left[W^c_w(N,\lambda,\phi)\right]^2} \\ & -\sqrt{\left[U_u(0,\lambda,\phi)\right]^2 + \left[U_v(0,\lambda,\phi)\right]^2 + \left[U_w(0,\lambda,\phi)\right]^2}\end{aligned}} \tag{93}$$

But again an approximation of Δg_{cl} in terms of the disturbing potential T is possible. With

$$\left|\nabla U(0)\right| \approx \left|\nabla U(N)\right| - N \cdot \left.\frac{\partial |\nabla U|}{\partial h}\right|_{h=0} \tag{94}$$

we get for eq. (30)

$$\Delta g_{cl}(\lambda,\phi) \approx \left|\nabla W^c(N,\lambda,\phi)\right| - \left|\nabla U(N,\phi)\right| + N \cdot \left.\frac{\partial |\nabla U(\phi)|}{\partial h}\right|_{h=0} \tag{95}$$

With eq. (28) for the downward continuation of the gravity disturbance to $h = N$ and eq. (16) for the normal gravity we can write it in the more usual form

$$\Delta g_{cl}(\lambda,\phi) \approx \delta g^c(N,\lambda,\phi) + N \cdot \left.\frac{\partial \gamma(\phi)}{\partial h}\right|_{h=0} \tag{96}$$

With the approximations (91) for δg^c and (56) for N we get

$$\Delta g_{cl}(\lambda,\phi) \approx \left.\frac{\partial T^c(h,\lambda,\phi)}{\partial h}\right|_{h=N} + \frac{T^c(N_1^c,\lambda,\phi)}{\gamma(0,\phi)} \cdot \left.\frac{\partial \gamma(\phi)}{\partial h}\right|_{h=0} \tag{97}$$

to calculate the classical gravity anomaly from the disturbing potential. Without previous knowledge of N, i.e. using eq. (55) instead of (56), we can approximate the classical gravity anomaly to:

$$\boxed{\Delta g_{cl}(\lambda,\phi) \approx -\left.\frac{\partial T^c(h,\lambda,\phi)}{\partial h}\right|_{h=0} + \frac{T^c(0,\lambda,\phi)}{\gamma(0,\phi)} \cdot \left.\frac{\partial \gamma(\phi)}{\partial h}\right|_{h=0}} \tag{98}$$

If we again replace the ellipsoidal normal by the radial direction and approximate additionally the normal gravity γ by its spherical term

$$\gamma(\phi) \approx \frac{GM}{r^2(\phi)} \quad \text{and} \quad \frac{\partial \gamma(\phi)}{\partial r} \approx -\frac{2\,GM}{r^3(\phi)} \tag{99}$$

(G is the gravitational constant and M is the mass of the Earth) we get the spherical approximation Δg_{sa} of the classical gravity anomaly to:

$$\boxed{\Delta g_{cl}(\lambda,\phi) \approx \Delta g_{sa}(\lambda,\phi) = -\left.\frac{\partial T^c}{\partial r}\right|_{h=0} - \frac{2}{r(\phi)}\, T^c(0,\lambda,\phi)} \tag{100}$$

where $r = r(\phi)$ is the distance to the centre of the coordinate system (spherical coordinate) of a point on the ellipsoid.

3.5.2 The Modern Gravity Anomaly

The calculation of Δg using eq. (31) at a given point (h,λ,ϕ) is possible if the height anomaly ζ has been calculated beforehand:

$$\boxed{\boxed{\begin{aligned}\Delta g(h,\lambda,\phi) = & \sqrt{\left[W_u(h,\lambda,\phi)\right]^2 + \left[W_v(h,\lambda,\phi)\right]^2 + \left[W_w(h,\lambda,\phi)\right]^2} \\ & -\sqrt{\left[U_u(h-\zeta,\lambda,\phi)\right]^2 + \left[U_v(h-\zeta,\lambda,\phi)\right]^2 + \left[U_w(h-\zeta,\lambda,\phi)\right]^2}\end{aligned}}} \tag{101}$$

To calculate it in terms of the disturbing potential analogously to the classical gravity anomaly we get:

$$\Delta g(h,\lambda,\phi) \approx \delta g(h,\lambda,\phi) + \zeta \cdot \left.\frac{\partial \gamma(\phi)}{\partial h}\right|_{h=0}, \quad \text{for} \quad h \geq h_t \tag{102}$$

Alike eq. (98) the approximation without knowledge of ζ (i.e. $\zeta = 0$) is:

$$\boxed{\Delta g(h,\lambda,\phi) \approx -\left.\frac{\partial T^c(h,\lambda,\phi)}{\partial h}\right|_{h} + \frac{T^c(0,\lambda,\phi)}{\gamma(0,\phi)} \cdot \left.\frac{\partial \gamma(\phi)}{\partial h}\right|_{h}} \tag{103}$$

which is valid for arbitrary points outside the geoid because T^c has been used (remember that $T^c = T$ for $h \geq h_t$). For $h = 0$ eq. (103) is the same as eq. (98) for the classical gravity anomaly Δg_{cl}, so the spherical approximation using eq. (99) is the same too:

$$\boxed{\Delta g(\lambda, \phi) \approx \Delta g_{sa}(\lambda, \phi) = -\left.\frac{\partial T^c}{\partial r}\right|_{h=0} - \frac{2}{r(\phi)}\, T^c(0, \lambda, \phi)} \tag{104}$$

3.5.3 The Topography-Reduced Gravity Anomaly

If we know the density $\rho(h, \lambda, \phi)$ of the masses above the geoid and the height $H(\lambda, \phi)$ of the topography above the geoid the calculation of the potential $V_t(h, \lambda, \phi)$ (and its derivatives) in eqs. (33) and (34) is in principle possible by numerical integration. But, however, it is extensive. Therefore, in the past, without today's computer power, the question was: how can the potential V_t of the topographical masses be replaced by a first approximation which results in a simple formula depending only on a constant density and the height $H(\lambda, \phi)$ of the point where it should be calculated?

As a simple but useful approximation for the topography-reduced gravity anomaly, the Bouguer anomaly Δg_B has been introduced as:

$$\Delta g_t r(\lambda, \phi) \approx \Delta g_B(\lambda, \phi) = \left|\nabla W^c(N, \lambda, \phi)\right| - A_B - \left|\nabla U(0, \phi)\right| \tag{105}$$

where

$$A_B(\lambda, \phi) = 2\pi G \rho H(\lambda, \phi) \tag{106}$$

is the attraction of the so called "Bouguer plate", which is a plate of thickness H (topographical height above geoid), constant density ρ and infinite horizontal extent (e.g. Hofmann-Wellenhof & Moritz, 2005). With the classical gravity anomaly (eq. 30) we get:

$$\Delta g_B(\lambda, \phi) = \Delta g_{cl}(\lambda, \phi) - 2\pi G \rho H(\lambda, \phi) \tag{107}$$

Unfortunately this cannot be expressed in terms of potentials because the potential of an infinite plate makes no sense (cf. the discussion in Vaníček et al., 2001 and Vaníček et al., 2004). The obvious idea to use the potential of a spherical shell as in section 3.1 results in a contribution of $-4\pi G \rho H$ which is twice the "Bouguer plate"-attraction. The plausible explanation is that the contribution of the far zone of the spherical shell, the whole opposite half sphere, cannot be neglected here, whereas in eq. (59), for the *difference* $V_t(N) - V_t^c(N)$ (which results in eq. 69), it can.

To find a simple approximation for V_t which is consistent with the results for the "Bouguer plate" one could define the potential of a spherical cap of constant thickness H, or a gaussian bell shaped "mountain" with height H, and an extend which produce the attraction of $A_B = 2\pi G \rho H$.

4 Calculation from Spherical Harmonics

4.1 Spherical Harmonics and the Gravity Field

The solid spherical harmonics are an orthogonal set of solutions of the Laplace equation represented in a system of spherical coordinates. (e.g. Hobson, 1931; Freeden, 1985; Hofmann-Wellenhof & Moritz, 2005). Thus, each harmonic potential, i.e. such which fulfils Laplace's equation, can be expanded into solid spherical harmonics. For this reason the stationary part of the Earth's gravitational potential W_a (the attraction part only, see eq. 5) at any point (r, λ, φ) on and above the Earth's surface is expressed on a global scale conveniently by summing up over degree and order of a spherical harmonic expansion. The spherical harmonic (or Stokes') coefficients represent in the spectral domain the global structure and irregularities of the geopotential field or, speaking more generally, of the gravity field of the Earth. The equation relating the spatial and spectral domains of the geopotential is as follows:

$$W_a(r,\lambda,\varphi) = \frac{GM}{r}\sum_{\ell=0}^{\ell_{\max}}\sum_{m=0}^{\ell}\left(\frac{R}{r}\right)^{\ell} P_{\ell m}(\sin\varphi)\left(C_{\ell m}^W \cos m\lambda + S_{\ell m}^W \sin m\lambda\right) \tag{108}$$

which shows the $1/r$-behaviour for $r \to \infty$, or written in the form

$$W_a(r,\lambda,\varphi) = \frac{GM}{R}\sum_{\ell=0}^{\ell_{\max}}\sum_{m=0}^{\ell}\left(\frac{R}{r}\right)^{\ell+1} P_{\ell m}(\sin\varphi)\left(C_{\ell m}^W \cos m\lambda + S_{\ell m}^W \sin m\lambda\right)$$

which is sometimes useful in practice. The notations are:

r, λ, φ	-	spherical geocentric coordinates of computation point (radius, latitude, longitude)
R	-	reference radius
GM	-	product of gravitational constant and mass of the Earth
ℓ, m	-	degree, order of spherical harmonic
$P_{\ell m}$	-	fully normalised Lengendre functions
$C_{\ell m}^W, S_{\ell m}^W$	-	Stokes' coefficients (fully normalised)

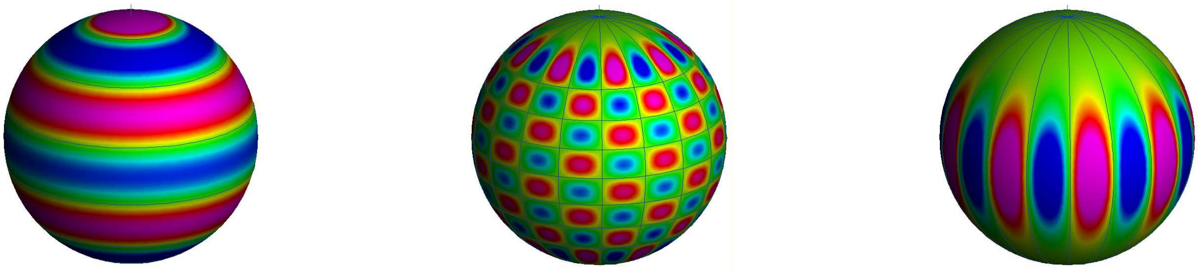
The transformation formulas between ellipsoidal (h, λ, ϕ) and spherical (r, λ, φ) coordinates can be found e.g. in (Hofmann-Wellenhof & Moritz, 2005). A spherical harmonic approximation of the gravity field up to a maximum degree ℓ_{max} (a so-called "gravity field model") consists of $(\ell_{max}+1)^2$ coefficients and the 2 values for GM and R to which the coefficients relate. The reference radius R of the expansion has only mathematical meaning. As can be seen from eq. (108), the product $C_{00}^W \times GM$ represents the gravitational constant times the mass of the Earth associated with the model. This means that C_{00}^W scales the formal value of GM which is given with the model. Usually C_{00}^W is defined to 1 to preserve the meaning of GM which itself is not separated into its two single values G (gravitational constant) and M (mass of the Earth) because it is known as product with a much higher accuracy than the two separate values. The degree 1 spherical harmonic coefficients $(C_{10}^W, C_{11}^W, S_{11}^W)$ are related to the geocentre coordinates and are zero if the coordinate systems' origin coincides with the geocentre. The coefficients C_{21}^W and S_{21}^W are connected to the mean rotational pole position.

Thus, eq. (108) represents the Earth's gravity field with an accuracy depending on the accuracy of the coefficients $(C_{\ell m}^W, S_{\ell m}^W)$ and a spatial resolution depending on the maximum degree ℓ_{max}. At a given point in space the difference of the real potential and the potential represented by the spherical harmonic expansion in eq. (108) depends on both, the coefficient's accuracy and the maximum degree ℓ_{max} of the expansion.

Equation (108) contains the upward-continuation of the gravitational potential from the Earth's surface for $r > r_{topo}$ and reflects the attenuation of the signal with altitude through the factor $(R/r)^{\ell}$. For points lying inside the Earth the spherical harmonic expansion gives the harmonic downward continuation W_a^c of the potential in a natural way simply by evaluating it for $r < r_{topo}$. However, possible singularities of this downward continuation (see the remarks in subsection 2.1) would result in divergence of the spherical harmonic series at the singular points for $\ell_{max} \to \infty$ (cf. the discussion of this topic for the Earth surface in Moritz, 1989). In practise ℓ_{max} is finite and the series can be evaluated, in principle, also for points lying inside the Earth $(r < r_{topo})$. However, the harmonic downward continuation, from its physical nature, is an unstable and ill-posed problem. That means the amplitudes of spatial undulations of the potential are amplified with depth (up to infinity at the locations of the singularities) and the amplification is bigger the shorter the wavelength of the undulation is. Mathematically this is obvious from the factor $(R/r)^{\ell}$ in eq. (108) for decreasing radius r and increasing degree ℓ. Thus, downward continuation in practise is always a (frequency-dependent) amplification of errors, i.e. in case of spherical harmonic representation an ℓ-dependent amplification of the errors of the coefficients $C_{\ell m}^W, S_{\ell m}^W$.

Note, that the spherical harmonics are calculated using spherical coordinates, so $r_{topo} = r_{topo}(\lambda, \varphi)$ is the distance of the point on the topography (Earth's surface) from the Earth's centre and φ is the spherical latitude to be distinguished from the ellipsoidal latitude ϕ.

Figure 4 presents examples for the three different kinds of spherical harmonics $P_{\ell m}(\sin\varphi) \cdot \cos m\lambda$: (a) zonal with $l \neq 0, m = 0$, (b) tesseral with $l \neq 0, m \neq l \neq 0$ and (c) sectorial harmonics with $\ell = m \neq 0$.



zonal: $\ell = 6$, $m = 0$ tesseral: $\ell = 16$, $m = 9$ sectorial: $\ell = 9$, $m = 9$

Figure 4: Examples for spherical harmonics $P_{\ell m}(\sin\varphi) \cdot \cos m\lambda$ [from -1 (blue) to $+1$ (violet)]

Obviously, the attraction part U_a of the normal potential U (see eq. 13), and thus the disturbing potential T, according to eq. (10), can be expanded into spherical harmonics too. If we denote the coefficients which represent U_a by $C^U_{\ell m}, S^U_{\ell m}$ the coefficients $C^T_{\ell m}, S^T_{\ell m}$ of the disturbing potential are simply the differences

$$C^T_{\ell m} = C^W_{\ell m} - C^U_{\ell m} \qquad \text{and} \qquad S^T_{\ell m} = S^W_{\ell m} - S^U_{\ell m} \tag{109}$$

The expansion of the ellipsoidal normal potential contains only terms for order $m = 0$ (rotational symmetry) and degree $\ell =$ even (equatorial symmetry). Recall that $S_{\ell,0}$ don't exist, so $S^T_{\ell,m} = S^W_{\ell,m}$. To calculate the normal potential in practise in most cases it is sufficient to consider only the coefficients $C^U_{00}, C^U_{20}, C^U_{40}, C^U_{60}$ and sometimes C^U_{80}. The disturbing potential T in spherical harmonics is:

$$T(r, \lambda, \varphi) = \frac{GM}{r} \sum_{\ell=0}^{\ell_{\max}} \left(\frac{R}{r}\right)^{\ell} \sum_{m=0}^{\ell} P_{\ell m}(\sin\varphi) \left(C^T_{\ell m} \cos m\lambda + S^T_{\ell m} \sin m\lambda\right) \tag{110}$$

for $r \geq r_{topo}$. For $r < r_{topo}$ equation (110) gives T^c, the harmonic downward continuation of T, introduced in section (3.1).

Here, we implicitly postulated that $C^U_{\ell m}$, the coefficients of the attraction part of the normal potential, and $C^W_{\ell m}$ and $S^W_{\ell m}$, the coefficients of the real potential (or the potential of a model approximating the real potential), are given with respect to the same values for GM and R. Usually this is not the case in practise where the normal potential coefficients $\hat{C}^U_{\ell m}$ are given with respect to separately defined values GM^U and R^U. From comparing the summands of the series separately, which must be equal due to orthogonality, the relation between them is found to be:

$$C^U_{\ell m} = \hat{C}^U_{\ell m} \times \frac{GM^U}{GM} \left(\frac{R^U}{R}\right)^{\ell} \tag{111}$$

and must be considered in eq. (109).

Each representation of a function in spherical harmonics like eq. (108) with an upper limit of summation $\ell_{max} < \infty$ corresponds to a low pass filtering, and ℓ_{max} correlates to the spatial resolution at the Earth surface. A usual simple estimation of the smallest representable feature of the gravity field, in other words, the shortest half-wavelength ψ_{min} (as spherical distance), that can be resolved by the $(\ell_{max} + 1)^2$ parameters $C_{\ell m}, S_{\ell m}$ is:

$$\psi_{min}(\ell_{max}) \approx \frac{\pi R}{\ell_{max}} \tag{112}$$

This estimation is based on the number of possible zeros along the equator.

At this point let us recall that the resolution of spherical harmonics is uniform on the sphere. This follows from the known fact that under rotation, a spherical harmonic of degree ℓ is transformed into a linear combination of spherical harmonics of the same degree. To illustrate it, imagine a single pulse somewhere on the sphere represented (as narrow as possible) by spherical harmonics up to a maximum degree and order ℓ_{max}. A rotation of the coordinate system will not change the shape of the pulse which means uniform resolution. Hence, a better estimation of $\psi_{min}(\ell_{max})$ seems to be the following: If we divide the surface of the sphere, i.e. $4\pi R^2$, into as many equiareal pieces A_{min} as the number of spherical harmonic coefficients, i.e. $(\ell_{max}+1)^2$, then the size of each piece is:

$$A_{min}(\ell_{max}) = \frac{4\pi R^2}{(\ell_{max}+1)^2} \tag{113}$$

The diameter of a spherical cap of this size is (in units of spherical distance):

$$\psi_{min}(\ell_{max}) = 4 \arcsin\left(\frac{1}{\ell_{max}+1}\right) \tag{114}$$

which characterise the size of the smallest bump, half-wavelength, which can be produced by $(\ell_{max}+1)^2$ parameters. For some selected maximum degrees the resolutions are given in Table 1. To demonstrate

Table 1: Examples of spatial resolution of spherical harmonics in terms of the diameter ψ_{min} of the smallest representable shape (bump or hollow) after eqs. (112) and (114)

Maximum Degree	Number of Coefficients	Resolution ψ_{min}			
		eq. (112)		eq. (114)	
ℓ_{max}	N	[degree]	[km]	[degree]	[km]
2	9	90.0	10000.000	77.885	8653.876
5	36	36.0	4000.000	38.376	4264.030
10	121	18.0	2000.000	20.864	2318.182
15	256	12.0	1333.333	14.333	1592.587
30	961	6.0	666.667	7.394	821.587
36	1369	5.0	555.556	6.195	688.321
40	1681	4.5	500.000	5.590	621.154
45	2116	4.0	444.444	4.983	553.626
50	2601	3.6	400.000	4.494	499.342
75	5776	2.4	266.667	3.016	335.073
180	32761	1.0	111.111	1.266	140.690
360	130321	0.5	55.556	0.635	70.540

how the resolution of spherical harmonics depends on the maximum degree ℓ_{max} of the development the following synthetic example has been constructed: A $(1° \times 1°)$-grid where all elements are zero except for two with the values 1 has been converted into spherical harmonic coefficients up to degree and order $\ell_{max} = 90$ using the numerical integration described in (Sneeuw, 1994). The two peaks are 6° (spherical distance) apart from each other. The cross-sections through the two peaks for different maximum degrees ℓ_{max} are shown in Figure (5). From Table (1) (eq. 114) one expects that a maximum degree of $\ell_{max} \approx 36$ suffices to resolve the peaks but the result of this example is that a slightly higher maximum degree of $\ell_{max} \approx 41$ is necessary.

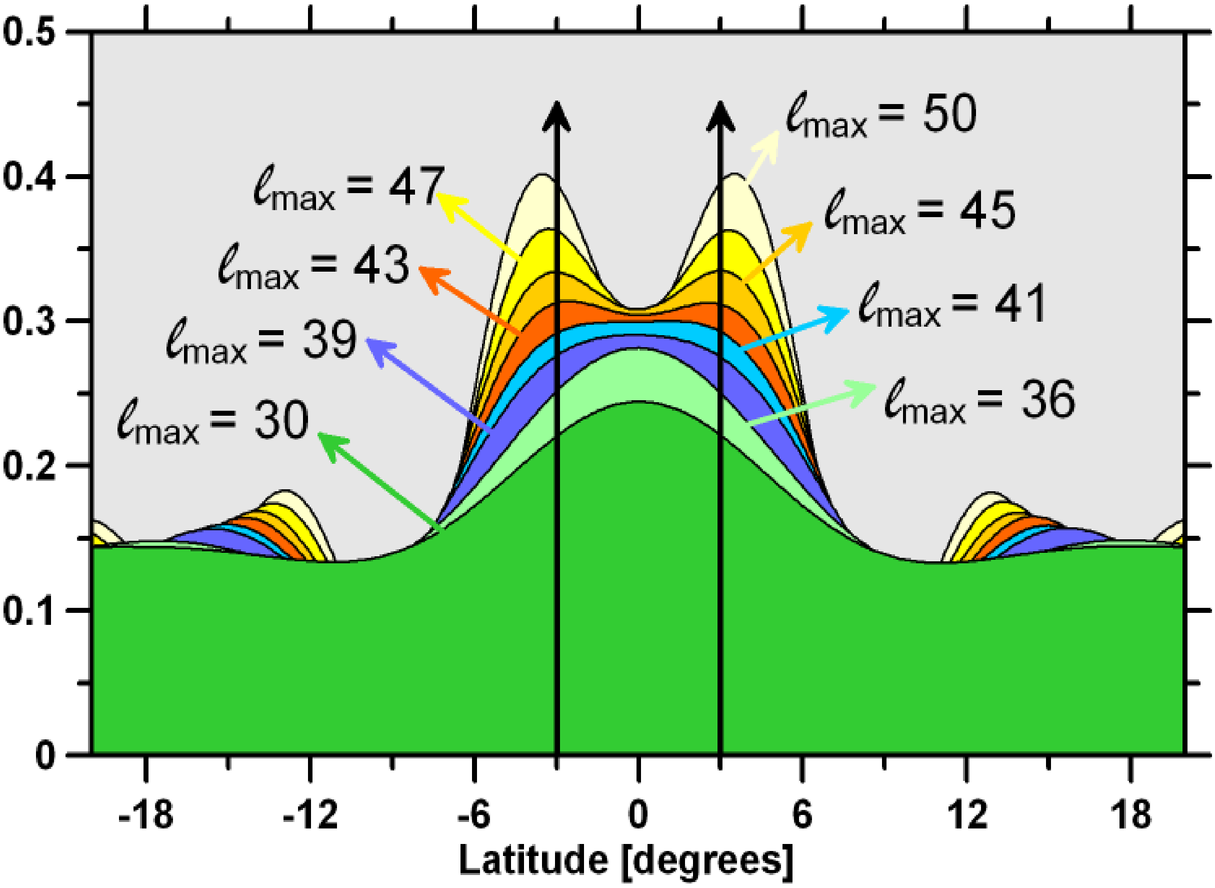


Figure 5: Cross-sections through 2 peaks, which are originally 6° apart, after approximation by spherical harmonics of different maximum degrees ℓ_{max}

4.2 The Geoid

To calculate the geoid undulation from eq. (70) or eq. (71) besides the potentials W and U, or $T = W - U$, a representation of the topography $H(\lambda, \varphi)$ must be available too. Usually the topography models are given as grids on the reference ellipsoid and have a much higher resolution (e.g. $2' \times 2'$) than the recent global gravity field models. To avoid adding parts of different resolution in eqs. (70) or (71), the topography model can also be transformed into a surface spherical harmonics expansion (a good summary for this technique can be found in Sneeuw, 1994):

$$H(\lambda, \varphi) = R \sum_{\ell=0}^{\ell_{\max}} \sum_{m=0}^{\ell} P_{\ell m}(\sin\varphi) \left(C_{\ell m}^{topo} \cos m\lambda + S_{\ell m}^{topo} \sin m\lambda \right) \tag{115}$$

where $C_{\ell m}^{topo}$ and $S_{\ell m}^{topo}$ are the coefficients of the expansion which are usually scaled by the reference radius R. Using the same upper limit ℓ_{max} of the expansion, the geoid N can be approximated according to eq. (70) by:

$$\boxed{\begin{aligned} N_1^s(\lambda,\varphi) = & \frac{GM}{r_e\ \gamma(r_e,\varphi)} \sum_{\ell=0}^{\ell_{\max}} \left(\frac{R}{r_e}\right)^{\ell} \sum_{m=0}^{\ell} P_{\ell m}(\sin\varphi) \left(C_{\ell m}^T \cos m\lambda + S_{\ell m}^T \sin m\lambda\right) \\ & -2\pi\, G\rho \left[R \sum_{\ell=0}^{\ell_{\max}} \sum_{m=0}^{\ell} P_{\ell m}(\sin\varphi) \left(C_{\ell m}^{topo} \cos m\lambda + S_{\ell m}^{topo} \sin m\lambda\right)\right]^2 \end{aligned}} \tag{116}$$

or according to eq. (71) by:

$$\boxed{\begin{aligned} \tilde{N}_2^s(\lambda,\varphi) = & \frac{GM}{r_\zeta\ \gamma(r_e,\varphi)} \sum_{\ell=0}^{\ell_{\max}} \left(\frac{R}{r_\zeta}\right)^{\ell} \sum_{m=0}^{\ell} P_{\ell m}(\sin\varphi) \left(C_{\ell m}^T \cos m\lambda + S_{\ell m}^T \sin m\lambda\right) \\ & -2\pi\, G\rho \left[R \sum_{\ell=0}^{\ell_{\max}} \sum_{m=0}^{\ell} P_{\ell m}(\sin\varphi) \left(C_{\ell m}^{topo} \cos m\lambda + S_{\ell m}^{topo} \sin m\lambda\right)\right]^2 \end{aligned}} \tag{117}$$

The radius-coordinate of the calculation point for the normal gravity γ is set to the latitude dependent radius-coordinate $r_e = r_e(\varphi)$ of points on the ellipsoid, and for the disturbing potential it is set to $r = r_e(\varphi)$ as well if eq. (70) is used, or to $r = r_\zeta(\lambda,\varphi)$ if eq. (71) is used, whereas in the latter case an approximation for $\zeta_e = \zeta_e(\lambda,\varphi)$ has to be calculated in a prior step using eq. (118).

4.3 The Height Anomaly

The calculation of the height anomalies $\zeta(\varphi,\lambda)$ from spherical harmonic potential models according to the iteration (eq. 82) is possible using eqs. (4), (5) and (108). Using the coefficients of the disturbing potential (eqs. 109, 110 and 111) the calculation according to eq. (77) or (79) is simple. Without using a topography model they can be calculated from eq. (79) by

$$\boxed{\zeta_{e1}(\lambda,\varphi) = \frac{GM}{r_e\ \gamma(r_e,\varphi)} \sum_{\ell=0}^{\ell_{\max}} \left(\frac{R}{r_e}\right)^{\ell} \sum_{m=0}^{\ell} P_{\ell m}(\sin\varphi) \left(C_{\ell m}^T \cos m\lambda + S_{\ell m}^T \sin m\lambda\right)} \tag{118}$$

and the radius-coordinate $r(\lambda,\varphi)$ of the calculation point is be set to $r_e = r_e(\varphi)$. With eq. (81) they can be calculated more accurately by

$$\boxed{\tilde{\zeta}_1 = \zeta_{e1} + \frac{H(\varphi,\lambda) + N(\varphi,\lambda)}{\gamma(0,\varphi)} \cdot \left.\frac{\partial T^c}{\partial r}\right|_{r=r_e}} \tag{119}$$

which can be calculated from spherical harmonics if we use eq. (115) for H and eq. (116) or (117) for N, and eqs. (92) and (125) for the radial derivative of T^c.

4.4 The Gravity Disturbance

To calculate the gravity disturbance from eq. (87) the gradients ∇W from eq. (85) and ∇U from eq. (86) have to be calculated from spherical harmonics. The gradient ∇W in spherical coordinates is (e.g. Bronshtein et al., 2004):

$$\nabla W = W_r \vec{e}_r + \frac{1}{r\cos\varphi} W_\lambda \vec{e}_\lambda + \frac{1}{r} W_\varphi \vec{e}_\varphi \tag{120}$$

where $W_r, W_\lambda, W_\varphi$ are the partial derivatives and $\vec{e}_r$, $\vec{e}_\lambda$, $\vec{e}_\varphi$ are the unit vectors pointing in the direction of r, λ and φ respectively. Consequently for $|\nabla W|$, considering the centrifugal potential Φ (eqs. 4 and 5), we have:

$$\boxed{|\nabla W| = \sqrt{[W_{ar} + \Phi_r]^2 + \left[\frac{1}{r\cos\varphi}(W_{a\lambda} + \Phi_\lambda)\right]^2 + \left[\frac{1}{r}(W_{a\varphi} + \Phi_\varphi)\right]^2}} \tag{121}$$

The derivatives of eq. (108) in spherical harmonics are:

$$\boxed{\begin{aligned} W_{ar} &= -\frac{GM}{r^2} \sum_{\ell=0}^{\ell_{\max}} \left(\frac{R}{r}\right)^\ell (\ell+1) \sum_{m=0}^{\ell} P_{\ell m}(\sin\varphi) \left(C_{\ell m}^W \cos m\lambda + S_{\ell m}^W \sin m\lambda\right) \\ W_{a\lambda} &= \frac{GM}{r} \sum_{\ell=0}^{\ell_{\max}} \left(\frac{R}{r}\right)^\ell \sum_{m=0}^{\ell} m\, P_{\ell m}(\sin\varphi) \left(S_{\ell m} \cos m\lambda - C_{\ell m}^W \sin m\lambda\right) \\ W_{a\varphi} &= \frac{GM}{r} \sum_{\ell=0}^{\ell_{\max}} \left(\frac{R}{r}\right)^\ell \sum_{m=0}^{\ell} \frac{\partial P_{\ell m}(sin\varphi)}{\partial\varphi} \left(C_{\ell m}^W \cos m\lambda + S_{\ell m}^W \sin m\lambda\right) \end{aligned}} \tag{122}$$

The centrifugal potential (eq. 4) in spherical coordinates is

$$\Phi = \frac{1}{2}\,\omega^2 r^2 (\cos\varphi)^2 \tag{123}$$

and its derivatives are:

$$\boxed{\Phi_r = \omega^2 r\,(\cos\varphi)^2 \quad , \quad \Phi_\lambda = 0 \quad , \quad \Phi_\varphi = -\omega^2 r^2 \cos\varphi\,\sin\varphi} \tag{124}$$

hence $|\nabla W|$ can be calculated from eq. (121) with eqs. (122) and (124).

The same formulas hold for calculating $|\nabla U|$ by replacing W by U and $C_{\ell m}^W$, $S_{\ell m}^W$ by $C_{\ell m}^U$, $S_{\ell m}^U$, thus the gravity disturbance δg outside the masses or its downward continuation δg^c can be calculated exactly for arbitrary points (r, λ, φ) in space from the spherical harmonic coefficients of a given gravity field model.

The spherical approximation δg_{sa} of the gravity disturbance (eq. 92), i.e. the radial derivative of the disturbing potential T, calculated from a spherical harmonic expansion of T is:

$$\boxed{\delta g_{sa}(r,\lambda,\varphi) = -\frac{GM}{r^2} \sum_{\ell=0}^{\ell_{\max}} \left(\frac{R}{r}\right)^\ell (\ell+1) \sum_{m=0}^{\ell} P_{\ell m}(\sin\varphi) \left(C_{\ell m}^T \cos m\lambda + S_{\ell m}^T \sin m\lambda\right)} \tag{125}$$

4.5 The Gravity Anomaly

Similar to the gravity disturbance δg, equations (121) to (124) can be used to calculate the gravity anomaly Δg exactly from spherical harmonics from eq. (101) (or from eq. (93) for the classical Δg_{cl}).

The spherical approximation Δg_{sa} (eq. 104) calculated from a spherical harmonic expansion of the disturbing potential T is:

$$\boxed{\Delta g_{sa}(r,\lambda,\varphi) = -\frac{GM}{r^2} \sum_{\ell=0}^{\ell_{\max}} \left(\frac{R}{r}\right)^\ell (\ell-1) \sum_{m=0}^{\ell} P_{\ell m}(\sin\varphi) \left(C_{\ell m}^T \cos m\lambda + S_{\ell m}^T \sin m\lambda\right)} \tag{126}$$

Nowadays this formula, as well as the spherical approximation of the gravity disturbance (eq. 125), is not accurate enough for most practical applications. Nevertheless the degree-dependent factors ($\ell + 1$ for δg_{sa} and $\ell - 1$ for Δg_{sa}) give theoretical insight into the different spectral behaviour and are useful for simulation studies.

5 Practical calculations using the model EIGEN-GL04C

The recent gravity field model EIGEN-GL04C (Förste et al., 2006) and the topography model ETOPO2 (U.S. Department of Commerce, 2001) have been used to calculate the described functionals and their approximations on global grids. All calculations are carried out with respect to the reference system WGS84. The differences between several approximations of one and the same functional are presented.

The topography model is necessary for two different purposes: (a) to calculate the exact coordinates on the Earth's surface for the height anomalies on the Earth's surface, the gravity disturbances and the modern gravity anomalies, and (b) to calculate the geoid undulations from pseudo height anomalies on the ellipsoid considering the topographical effect. For (a) bi-linear interpolation of the original ETOPO2-grid is used to calculate the positions as accurately as possible. For (b) the gridded values of the original topography model has been transformed into a (surface-) spherical harmonic expansion using the formulas described by Sneeuw (1994). The maximum degree of EIGEN-GL04C is $\ell_{max} = 360$, so the topography has been used with the same resolution.

5.1 Geoid and Height Anomaly

The convergence of the iterative calculation of the "downward continued" geoid $N^c(\lambda, \phi)$ from $W^c(\lambda, \phi)$ and U_0 using eq. (54) with N_1^c from eq. (55) or (79) as start value is very fast. For the calculation using spherical harmonics eqs. (108) and (118) are used for W^c and N_1^c respectively. With a 64-bit accuracy (real*8 in Fortran, which are about 16 decimal digits) convergence is reached after 2 or 3 steps (step 1 is from N_1^c to N_2^c in our terminology). The spatial distribution of the accuracy of the start value N_1^c from eq. (55) and the first iteration N_2^c of eq. (54) are demonstrated by figures (6) and (7). Figure (8) shows the accuracy of the (non-iterative) approximation $\tilde{N}_2^c$ of eq. (56). Thus, to calculate the real geoid N from the "downward continued" geoid N^c by adding the influence of the topography (see eq. 60) the calculation of N^c has to be carried out with the required accuracy. However, as mentioned in section (3.1), the influence of the topographic masses are the accuracy limiting factor.

If we start the iterative calculation of the height anomalies $\zeta(h_t, \lambda, \phi)$ (eq. 82) with the pseudo height anomaly on the ellipsoid ζ_{e1} according to eqs. (79) and (118) the mean difference between start values and convergence is about 30 times bigger than for the iterative calculation of N^c using eq. (54). Nevertheless the iteration converges after 3 or 4 steps. The differences between the start values ζ_{e1} and the convergence ζ are shown in figure (9) and the accuracy of the first iteration ζ_1 is shown in figure (10). Figure (11) shows the accuracy of the (non-iterative) approximation $\tilde{\zeta}_1$ of eqs. (81) and (119). Usually the chosen calculation formula or algorithm will be a compromise between computing effort and accuracy requirement.

Figure (12) shows the differences between the height anomalies on the ellipsoid from eqs. (79) and (118) and the geoid heights from eqs. (71) and (117) (calculated using the spherical shell approximation for the topography). The maximum exceeds 3 metres. Thus, to calculate geoid undulations in continental areas, the potential of the topographic masses must be approximated somehow at the geoid inside the masses. The difference between the potential of the topographic masses and its harmonic downward continuation, both evaluated on the geoid, cannot be neglected.

The weighted means, the minima and the maxima of the grid differences shown in figures (6) to (12) are summarised in table (2).

5.2 Gravity Disturbance and Gravity Anomaly

Figure 13 shows the differences $(\Delta g_{cl} - \Delta g)$ between the classically defined and the modern (Molodensky's theory) gravity anomalies. They range from -12 to $+24$ mgal. Therefore, when using gravity anomaly data sets which are derived from real measurements it is important to know how they are reduced.

The differences $(\delta g_{sa} - \delta g)$ between spherically approximated and exactly calculated gravity disturbances are nearly the same as $(\Delta g_{sa} - \Delta g)$, the differences between spherically approximated and exact modern (Molodensky's theory) gravity anomalies. The differences $(\Delta g_{sa} - \Delta g)$ are shown in figure 14.

Table 2: Differences between varying approximation levels of height anomalies and geoid undulations

Differences [m]	wrms [m]	min [m]	max [m]
$N_1^c - N^c$	$1.3 \cdot 10^{-3}$	$-1.5 \cdot 10^{-2}$	$2.2 \cdot 10^{-2}$
$N_2^c - N^c$	$1.4 \cdot 10^{-7}$	$-5.7 \cdot 10^{-6}$	$3.8 \cdot 10^{-6}$
$\tilde{N}_2^c - N^c$	$2.4 \cdot 10^{-4}$	$-1.8 \cdot 10^{-3}$	$5.0 \cdot 10^{-5}$
$\zeta_{e1} - \zeta$	$3.8 \cdot 10^{-2}$	$-3.5 \cdot 10^{-1}$	1.1
$\zeta_2 - \zeta$	$4.7 \cdot 10^{-5}$	$-3.0 \cdot 10^{-4}$	$2.0 \cdot 10^{-3}$
$\tilde{\zeta}_1 - \zeta$	$6.5 \cdot 10^{-3}$	$-1.1 \cdot 10^{-1}$	$8.5 \cdot 10^{-2}$
$\zeta_{e1} - \tilde{N}_2^s$	$2.4 \cdot 10^{-1}$	0.0	3.67

They range from -13 to $+23$ mgal. To compare or combine gravity anomalies or gravity disturbances derived from terrestrial measurements with those derived from gravity field models in spherical harmonic representation the spherical approximation is obviously not accurate enough.

The differences ($\Delta g_{sa} - \Delta g_{cl}$) between spherically approximated and classical gravity anomalies are smaller, they range from -1.48 to $+0.83$ mgal and are shown in figure 15. Hence, to calculate classical gravity anomalies the spherical approximation can be used if the highest accuracy is not required.

The weighted means, the minima and the maxima of the grid differences shown in figures (13) to (15) are summarised in table (3).

Table 3: Differences between the gravity anomalies, the classical gravity anomalies and the gravity anomalies in spherical approximation

Differences [mgal]	wrms [mgal]	min [mgal]	max [mgal]
$\Delta g_{cl} - \Delta g$	0.73	-12	26
$\Delta g_{sa} - \Delta g$	0.69	-13	23
$\Delta g_{sa} - \Delta g_{cl}$	0.11	-1.48	0.83

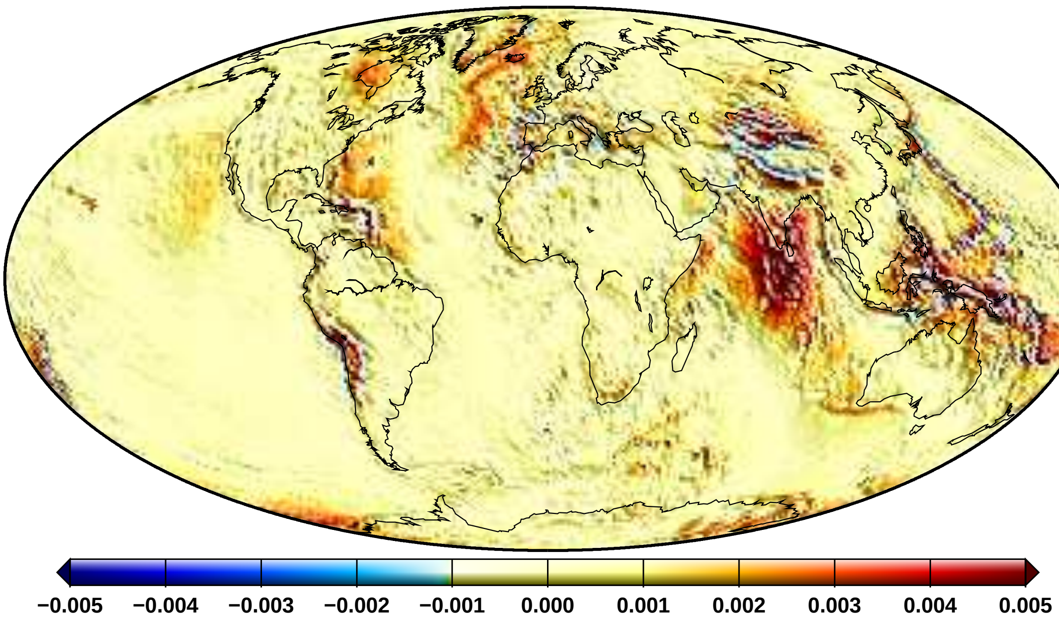


Figure 6: The difference between the start value N_1^c and the convergence N^c of eq. (54): $(N_1^c - N^c)$; wrms $= 1.3 \times 10^{-3}$ m, min $= -0.015$ m, max $= 0.022$ m

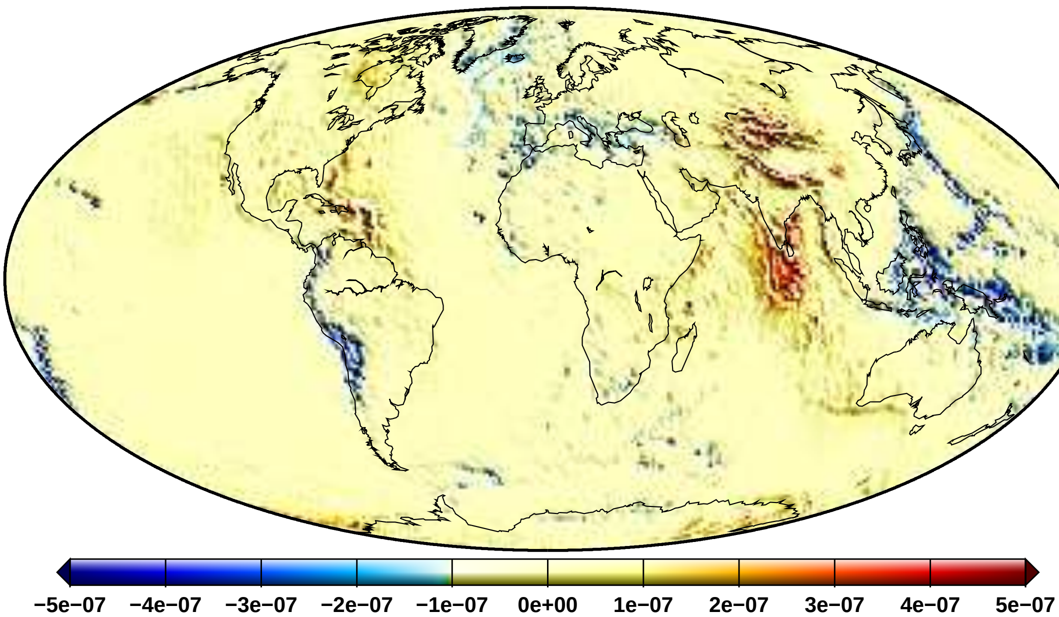


Figure 7: The difference between the first iteration N_2^c and the convergence of eq. (54): $(N_2^c - N^c)$; wrms $= 1.4 \times 10^{-7}$ m, min $= -5.7 \times 10^{-6}$ m, max $= 3.8 \times 10^{-6}$ m

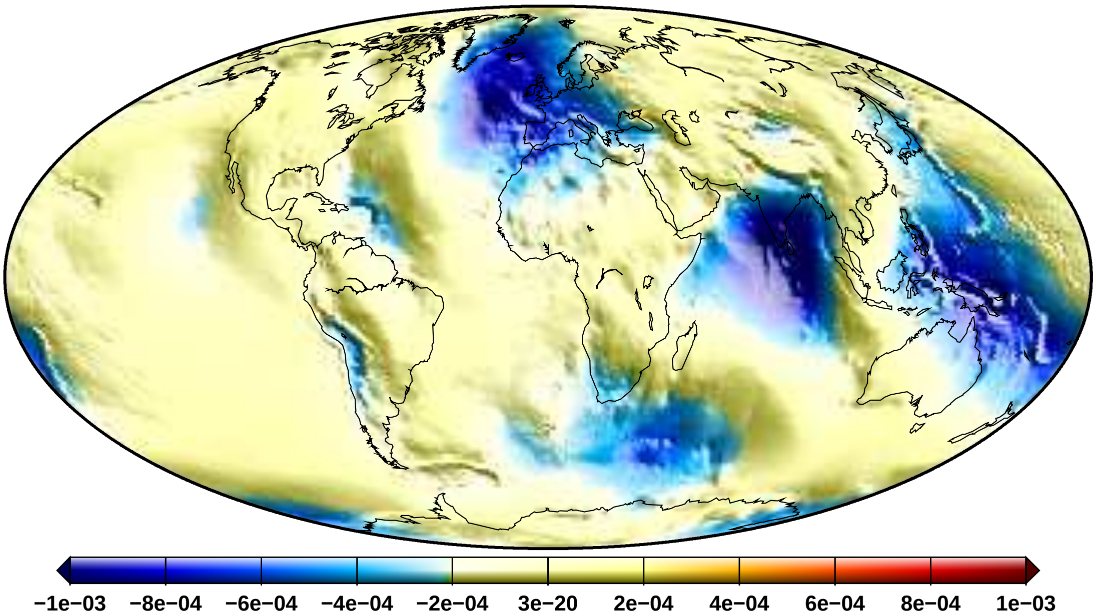


Figure 8: The difference between the approximation in eq. (56) and the convergence of eq. (54): $(\tilde{N}_2^c - N^c)$; wrms = 2.4×10^{-4} m, min = -1.8×10^{-3} m, max = 5×10^{-5} m

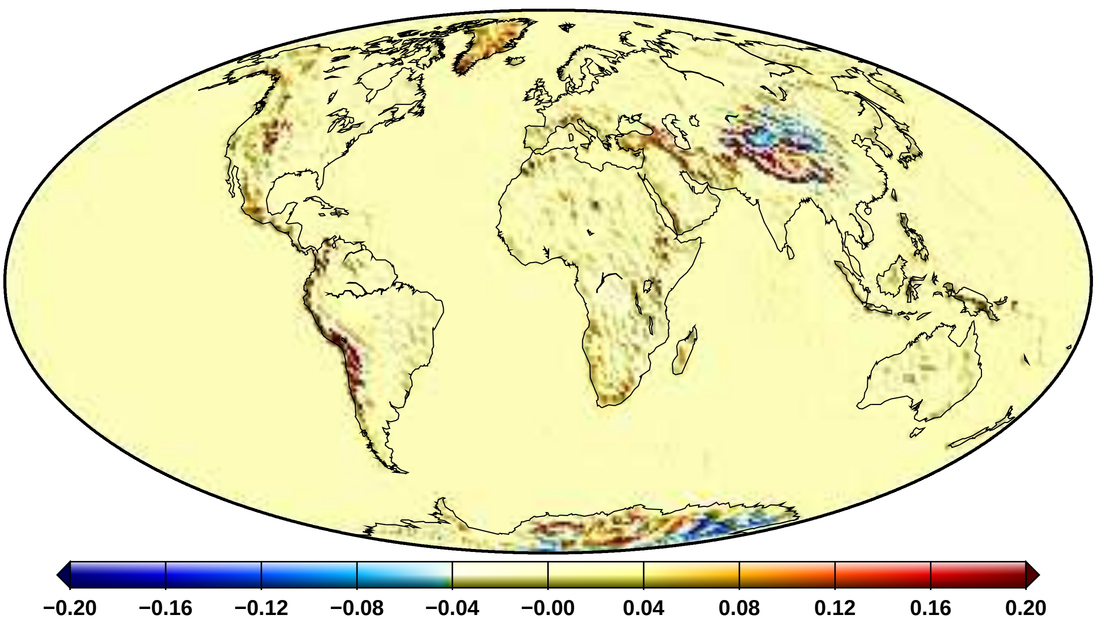


Figure 9: The difference between the pseudo height anomalies on the ellipsoid and the height anomalies on the Earth's surface: $(\zeta_{e1} - \zeta)$; wrms = 0.038 m, min = -0.35 m, max = 1.1 m

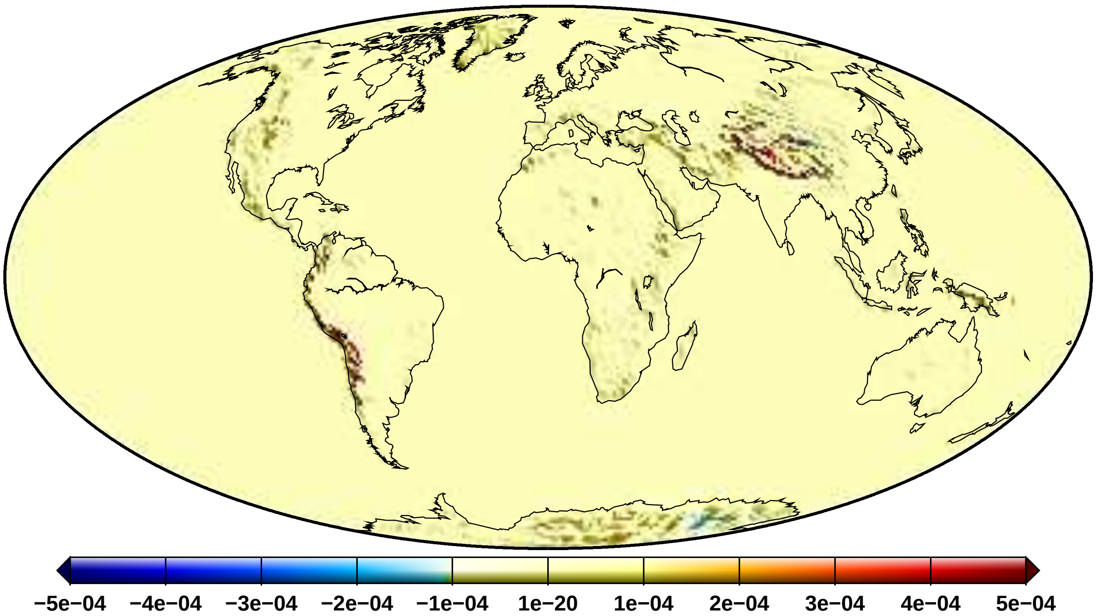


Figure 10: The differences between the first iteration ζ_2 and the convergence of eq. (82): $(\zeta_2 - \zeta)$; wrms $= 4.7 \times 10^{-5}$ m, min $= -3.0 \times 10^{-4}$ m, max $= 2.0 \times 10^{-3}$ m

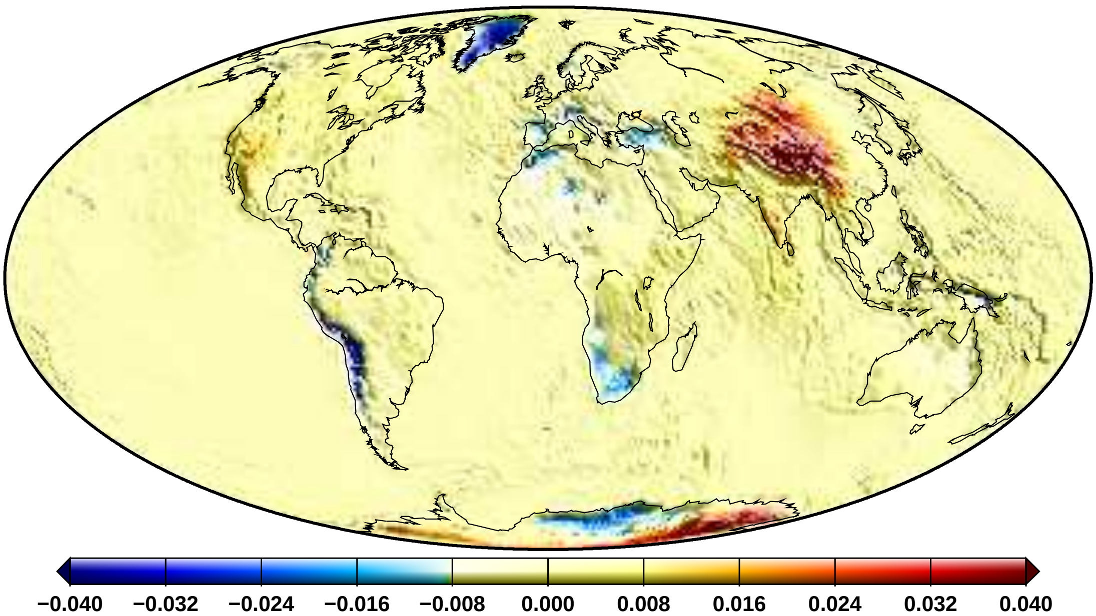


Figure 11: The difference between the approximation $\tilde{\zeta}_1$ of eq. (81) and the convergence of eq. (82): $(\tilde{\zeta}_1 - \zeta)$; wrms $= 6.5 \times 10^{-3}$ m, min $= -0.11$ m, max $= 0.085$ m

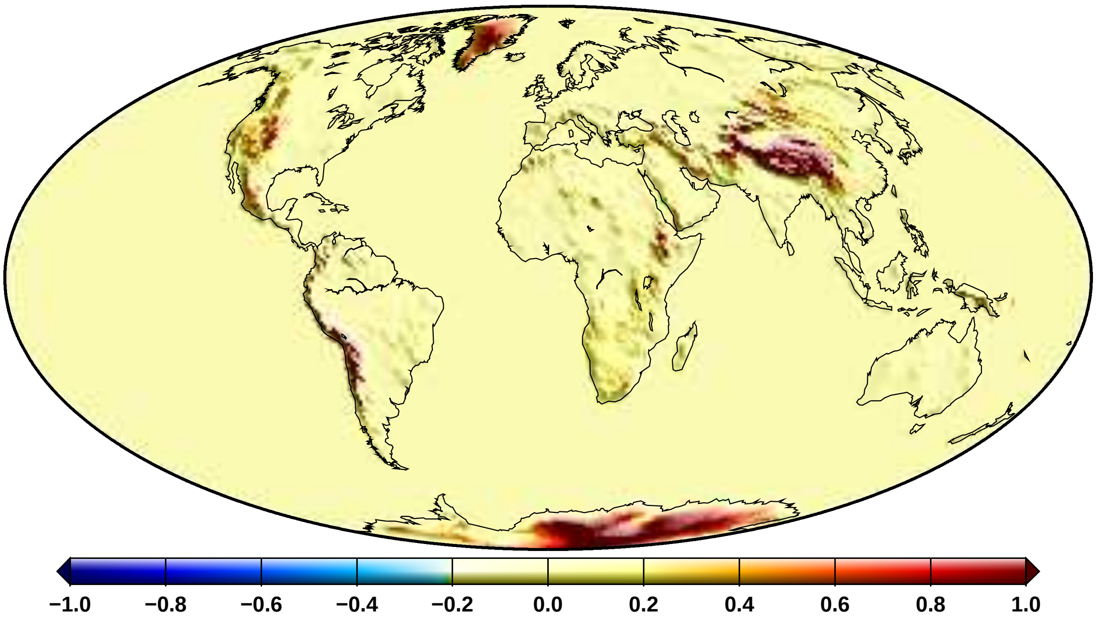


Figure 12: The difference between pseudo height anomalies on the ellipsoid of eq. (118) and geoid undulations from eq. (117): $(\zeta_{e1} - \tilde{N}_2^s)$; wrms $=$ 0.24 m, min $=$ 0.0 m, max $=$ 3.67 m

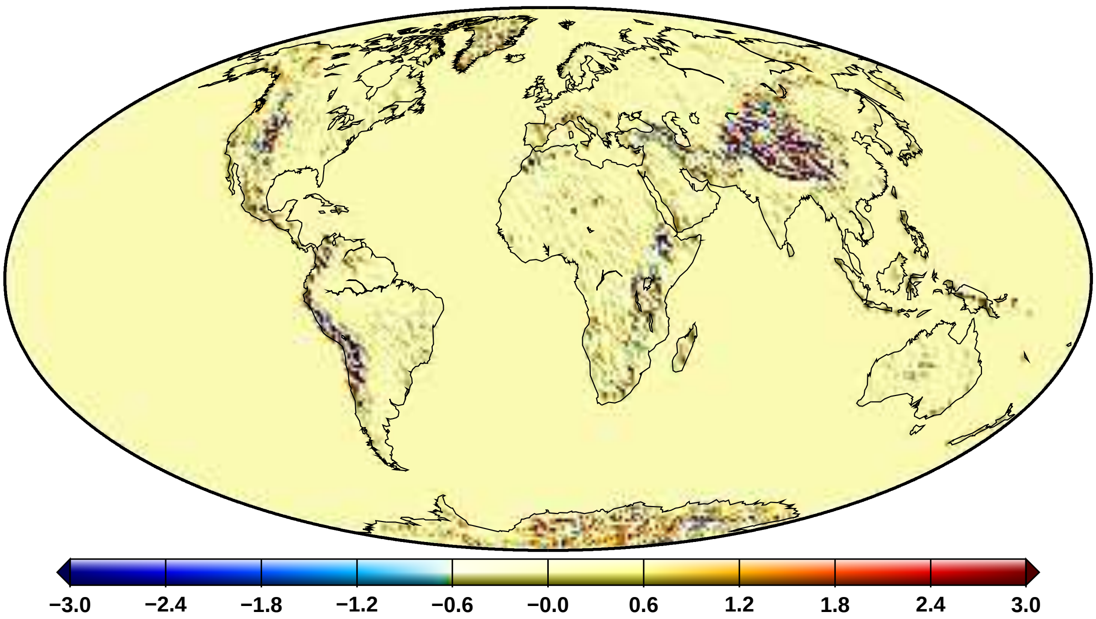


Figure 13: The difference between classical and modern (Molodensky's theory) gravity anomalies: $(\Delta g_{cl} - \Delta g)$; wrms $=$ 0.73 mgal, min $= -12$ mgal, max $=$ 26 mgal

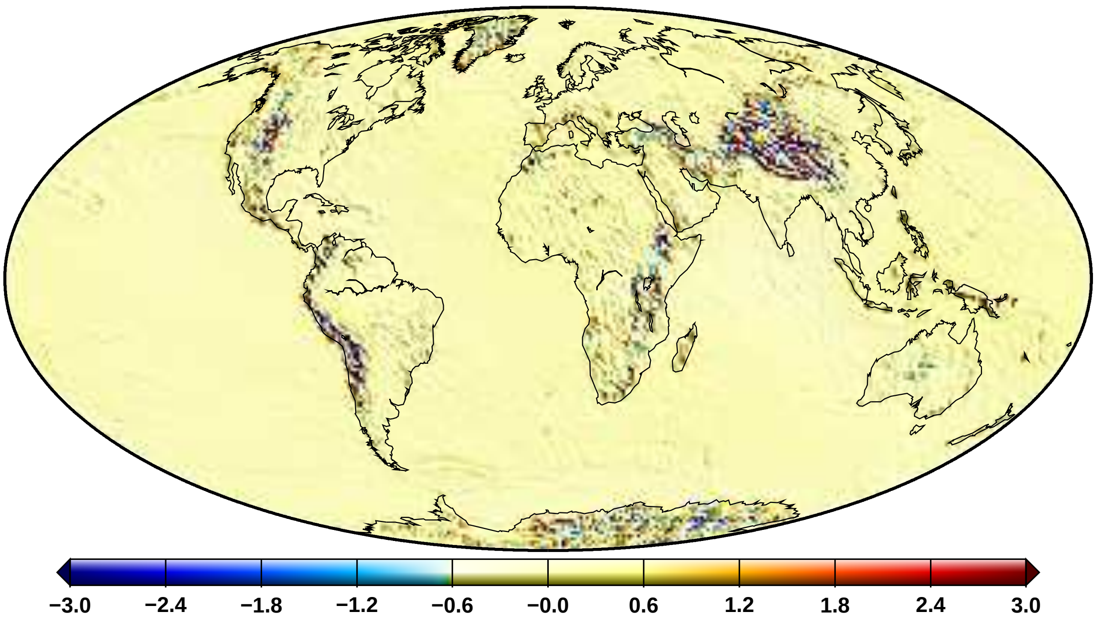


Figure 14: The difference ($\Delta g_{sa} - \Delta g$) between spherical approximated gravity anomalies and modern (Molodensky's theory) gravity anomalies; wrms = 0.69 mgal, min = −13 mgal, max = 23 mgal

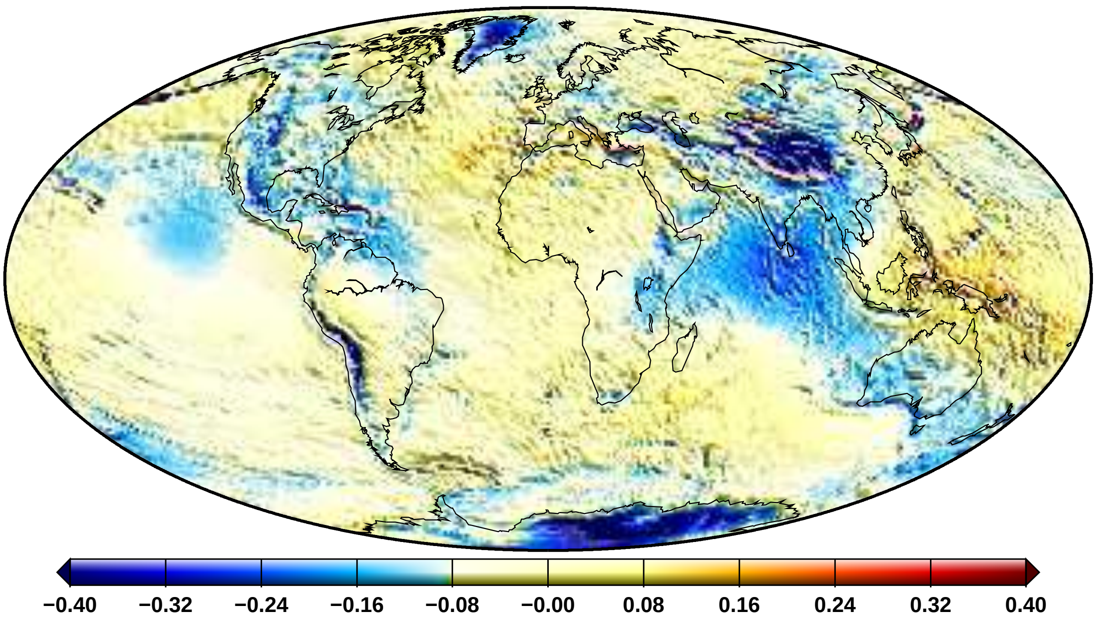


Figure 15: The difference ($\Delta g_{sa} - \Delta g_{cl}$) between spherical approximated and classical gravity anomalies; wrms = 0.11 mgal, min = −1.48 mgal, max = 0.83 mgal

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